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A GENERAL METHOD TO DETERMINE THE STABILITY OF COMPRESSIBLE FLOWS

Rolf Adolf Guenther
and I-Dee Chang

STANFORD UNIVERSITY
Department of Aeronautics and Astronautics
Stanford, California 94305

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In this report several problems were studied using two completely different approaches. The initial method was to use the standard linearized perturbation theory by finding the value of the individual small disturbance quantities based on the equations of motion. These were serially eliminated from the equations of motion to derive a single equation that governs the stability of fluid dynamic system. These equations could not be reduced unless the steady state variable depends only on one coordinate. The stability equation based on one dependent variable was found and was examined to determine the stability of a compressible swirling jet. The results obtained were very similar to those of Lallas for finite domains. There are differences that could be explained to slight changes in initial assumptions and also due to the fact that the interval of integration is all space rather than a finite subspace.

The second approach was that of using a Lagrangian approach to the problem. Since the equations developed were based on different assumptions, the condition of stability was compared only for the Rayleigh problem of a swirling flow, both examples reduce to the Rayleigh criterion. The second method allows including the viscous shear terms which is not possible in the first method. The same problem was again examined to see what effect shear has on stability.

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NOMENCLATURE

a	Speed of Sound.
$\vec{a}$	The incompressible convective acceleration vector, $(\vec{v} \cdot \Delta \vec{v})$.
$\vec{a}_i$	Basis vectors in reference coordinate system.
A	Compressible flow operator (2-13').
A_1, A_2	Coefficient of ψ used to determine the stability of a cylindrical shear layer.
$\vec{b}$	Rate of change of the perturbation displacement vector in the direction $\vec{v}$ $(\vec{v} \cdot \Delta \vec{\xi})$.
$\vec{b}_i$	Basis vectors of coordinate system tied to non-deformed material element.
B	The gradient of the convective perturbation operator in the r-direction, $\Omega DU + \ell D*(r\Omega)/r$.
c	The Rayleigh wave speed.
c^2	The speed of sound (1-4)
C	The vorticity or the velocity gradient used in defining the Richardson criterion, $DL/2 + 2\Omega \ell /r$.
C	Non-isentropic exponent for the pressure-density relationship.
$\vec{C}_i$	Basis vectors in the unperturbed deformed material element.
C_p	Specific heat at constant pressure.
$\vec{d}_i$	Basis vectors in the deformed perturbed material element.
ds	Differential length element.
dt	Differential volume element.
D	Derivative with respect to r, d/dr.
e	Square of the absolute wave number for cylindrical flow, $\alpha^2 + \ell^2/r^2$.

f	An arbitrary function.
F	An arbitrary scalar.
$F(\vec{\xi})$	The inviscid Lagrangian perturbation quantity required to determine the stability of the flow (Chapter 3).
$F'(\vec{\xi})$	The Lagrangian perturbation quantity for the full problem (Chapter 3).
g_{ij}	Metric tensor in the new coordinate system.
G	Rayleigh radial perturbation velocity function (Introduction).
h_{ij}	Metric tensor in the perturbed coordinate system.
$H_{\nu}^{(1)}$	Hermite polynomial of the first kind used to determine the stability of a compressible cylindrical shear layer.
i	The imaginary number, $\sqrt{-1}$.
$\underset{\sim}{I}$	The identity tensor.
J_{ν}	Bessel function of non-integer order.
K	Energy difference function, (3-10).
ℓ	Azimuthal wave number.
L	Operator formed by the substantial derivative acting on a perturbed quantity, $L = \omega + u_{\alpha} + \ell\Omega$.
L_I	The imaginary part of L.
L_R	The real part of L.
m	Perturbation mass flux in the radial direction, $\rho v'r$.
$\vec{m}$	Perturbation mass flux vector, (1-2).
$\vec{M}$	Mass flux vector (1-2).
M	Angular speed of sound squared over the radial distance $\Omega^2 r/a^2$
M	Mach number function, (Chapters 2 and 4).
n	Frequency.
$\vec{n}$	Vector normal to a surface or boundary.

N^2	Brunt-Vaisälä buoyancy frequency.
P	Steady state pressure.
q	Arbitrary flow field quantity.
q'	Arbitrary perturbed flow field quantity.
$\vec{q}$	Heat flux vector.
r	Radial coordinate.
$\vec{r}$	Arbitrary position vector or distance.
R	Reynolds number.
R	Perfect gas constant.
Re	Real part of an operator or quantity.
$\vec{R}_o$	Position in space at time t_o.
$\vec{R}^o$	Position in space at time t.
S	Entropy.
t	Time.
t_o	Initial time.
T	Temperature.
u'	Perturbed axial velocity component.
U	Axial velocity component.
V'	Radial perturbation velocity component.
$\vec{V}$	Velocity vector.
W	Circumferential velocity component.
$\vec{x}$	Arbitrary position vector in reference coordinate system.
$\vec{y}$	Arbitrary position vector in the coordinate system fixed with respect to a material element.

GREEK SYMBOLS

α Axial wave number.

β Mach number function that includes the wave speed (Chapters 2 and 4).

γ Ratio of specific heats.

δ A small length, thickness or value.

Δ_s Incremental change in value of a quantity across a discontinuity $\Delta_s f = f_{r + \epsilon} - f_{r - \epsilon}$.

ϵ A small value used in the definition of stability (Introduction).

η Normal component of the displacement vector.

θ Azimuthal coordinate.

λ Characteristic value.

μ Coefficient of viscosity.

ν $\sqrt{\ell^2 + 1}$.

$\vec{\xi}$ The perturbation displacement vector.

π Perturbation pressure (2-2).

ρ Density.

$\underset{\sim}{\tau}$ Shear stress tensor.

Ψ Transformed perturbation function, $m/L^{\frac{1}{2}}$.

ω Frequency.

Ω Curl $\vec{v}$ or angular rate w/r (2-6).

Ω_R Rayleigh discriminant, $2\Omega D^*(r\Omega)$.

SUPERSCRIPTS

1   Perturbed quantity.

$*$ Complex conjugate, adjoint or the derivative $D^* = d/dr + 1/r$.

SUBSCRIPTS

c   Critical layer or value.

I   Imaginary part.

R   Real part.

i, j   Indices referring to coordinate directions.

*   Derivative, $D_* = D - 1/r$.

INTRODUCTION

The basic concept of the stability of a particular flow is obtained
by observing the effect that the viscous, inertia and body forces have on a
fluid element as it proceeds through space and time. In general a flowing
medium will exhibit the smooth streamlines and steady motion of a laminar
flow for some time after it is set into motion. After a certain distance
the paths of the individual fluid elements may change into a chaotic
motion about a mean streamline. This type of motion can be observed in
flows of the boundary layer type as they progress through stages from laminar
to turbulent ones. Other types of changes may take place such as observed
by Taylor [1] and Görtler [2] in which the flow under the action of centri-
fugal forces does not become turbulent but changes to another form by the
appearance of vortices in the direction of motion. In each of these
the transition from one type of flow to another does not take place until
a certain set of circumstances is satisfied. Thus the steady flow remains
unchanged until the growth of the perturbations exceeds the inherent damping
of the flow field.

Since the transition from laminar to turbulent flow is not the only
possibility, a definition of stability must be written such that all limits
of stability may be included. To obtain such a definition we will resort to
the one used by Sattinger[3] in his paper on the "Mathematical Problem of
Hydrodynamic Stability".

Definition: Let $v(t)$ be a solution of the perturbed equation. The steady state flow is <u>stable</u> if in the limit as time approaches infinity the absolute value of the disturbance, $v(t)$, goes to zero whenever the perturbation at time $t = t_o$ is finite. The flow is <u>conditionally stable</u> if, for an $E > 0$ there is a $\delta > 0$ such that $|V(t)| < E$ for all $t \geq 0$ and in the limit as t approaches infinity $|V(t)| = 0$ whenever $|V(0)| < \delta$. The flow is unstable if this condition is not met.

Unconditional stability implies that the flow is stable regardless of the magnitude of the perturbation applied. This is similar to placing a ball at the bottom of an infinitely deep well in a gravitational field. Since the location of minimum potential energy is at the bottom of the well, the ball will always return to it. Conditional stability is shown by placing the ball in a well of finite depth. If the initial displacement is sufficiently large, the ball will leave the well and be free to find a new location of minimum potential energy. Instability results when any perturbation will move the ball in such a manner that it will not return to its initial point; i.e. placing the ball on a hill. In like manner the stability of a steady flow may be established via a more general energy approach in which the kinetic and potential energies of the system are defined by a more complicated set of forces and reactions. The problems in hydrodynamics are initial value problems because a perturbation is applied and its growth in time is considered. The stability so derived satisfies the definition for conditional stability. For most theories the solution of the stability boundaries is obtained via a set of equations that are linearized with respect to the perturbation variables. In this type of an analysis the velocity, pressure, density, etc. are perturbed and the linearized forms of the

equations are retained after substitution into the full viscous or inviscid equations. The non-linear perturbation terms are dropped because they occur as products of the linearized variables, and therefore, are assumed to be small. The argument that this type of solution describes the actual phenomenon is known as the "linearization hypothesis". The method is open to question because the derivatives of the perturbation quantities appear in the non-linear terms and these in general do not have to be as small as the perturbation quantities themselves. The wealth of experimental data, however, does tend to support this type of an analysis. The method has its limitations because the perturbations are assumed to infinitesimal; and therefore, the results do not depend on the amplitude of the disturbance. Thus it breaks down if the perturbations are of sufficient size so that the non-linear interactions take over and either limit further growth of the disturbance or amplify it disportionately.

As stated earlier the vast majority of analytical interpretations of stability were obtained via linearized analyses of stationary flows. The first accurate analysis of a stationary liquid jet was made by Lord Rayleigh[4] using an inviscid, incompressible linearized form of the perturbation equations. In such an analysis it is not assumed that the stationary flow is inviscid only that the effect of the viscous forces acting on the perturbation equations is of higher order and may be ignored. The work of Rayleigh was preceded by the experimental work of Savart[5] and the analytical work of Plateau[6]. Plateau was able to derive certain properties of the stability but he was unable to derive the most likely wavelength at which instability would occur. This was one of the contributions that was made by Rayleigh.

Rayleigh used the inviscid form of the perturbation equations and in the limit of high Reynolds numbers or vanishingly small viscosity it is expected that there will be no difference between the full and inviscid solutions. This is of course true in the bulk of the fluid. But for the inviscid fluid there is a critical layer at which the velocity of the disturbance and the steady flow are equal. If the distrubance velocity is complex, this layer lies in the complex plane. The problem of stability has by the linearization been reduced to an eigenvalue problem of the Orr-Sommerfeld kind. Rayleigh's equation became:

$$(u-c) \frac{d}{dr} \left[\frac{r}{n^2+a^2 r^2} \frac{d r G_i}{dr} \right] - (u-c)G_i - r G \frac{d}{dr} \left(\frac{r u'}{n^2 + a^2 r^2} \right) = 0$$

It can be seen immediately that letting u = c presents an immediate problem and results in a singularity for the solution G. The addition of the viscous terms renders the problem solvable because (u - c) is not a multiplier of the highest order derivative. Hence, the solution for G would remain finite. Physically the addition of a vanishingly small viscosity spreads the effect of the critical line over a finite width, $|r - r_c| < \delta$ that goes to zero as the viscosity goes to zero. If the value of r_c is real and non-zero, then the critical layer is a thin sheet in the fluid and the properties across the layer behave as they do in a two dimensional primary flow. This behavior was first noticed by Pretsch[7] and hence the effect of friction in a critical layer can be obtained by resorting to the known solution of two-dimensional primary flows. The two dimensional analyses were expanded by Tollmien[8] and Lin[9]. For amplified disturbances the effect of the friction is negligible for vanishing viscosity; and for neutral disturbances which also satisfy the inviscid equations of motion, the same branch

of the logarithm must be taken as for the slightly amplified case. Damped

solutions require integration of the above equation in the complex plane as

was shown by Lin[9]. Thus for damped solutions friction must always be

taken into account in certain regions. This was demonstrated by O. Tietjen[54]

for laminar, linear velocity profiles over a wall that would be stable for

inviscid flow. The addition of small viscosity to these profiles did not

lead to damping but led to amplification for all Reynolds numbers and wave-

lengths. Tollmien was able to explain the paradox by showing that the

viscosity for this case must be taken into account over the entire flow

especially in the neighborhood of the critical layer where the velocity

gradient becomes infinite in frictionless theory. The existence of viscos-

ity forces the velocity gradient to remain finite. The effect of the vis-

cosity could be evident only if the curvature profile was included and the

viscosity was included at the wall and at the critical layer.

The inviscid perturbation solutions of Lord Rayleigh are valid for

the straight jet and for rotational Couette flow because even for infinite

Reynolds numbers there exists a certain range of wavelengths that result

in instability. Therefore, he was able to state two important theorems

that relate to more general flows. The first became known as the Rayleigh

discriminant for inviscid rotational flow: "An inviscid rotating flow is

unstable if the square of its circulation decreases outward." The second

is the point-of-inflection criterion: "Velocity profiles that possess a

point of inflection are unstable". These and a final theorem that limits

the speed of propagation of an instability govern the so-called frictionless

instabilities because laminar flow subject to those conditions are unstable

even without considering the effect of viscosity on the oscillations of the motion. Viscosity does play a small role in either of the above theorems. For the first it smudges the boundaries somewhat and adds some damping to the system (Fig. 1). For the second, as discussed earlier, it shows that even flows without a point of inflection in the velocity profile tend to become unstable. The second theorem as initially stated was a necessary condition for the stability of certain flows, but Tollmien[10] was able to show much later that it was a sufficient condition also.

The stability condition as stated does not require that a flow go from a laminar to a turbulent to be unstable, only that the disturbance not die out with time. Examples of stationary flows that admit secondary solutions are vortices in Couette flow, Bénard cells and the Görtler vortices in concave parallel flows. These have been studied extensively. In each flow, the initial or primary flow bifurcates and a secondary stable solution appears that satisfies the same boundary conditions as the primary flow. Extensive mathematical work in the theory of partial differential equations has been done to provide rigor to the solution of the linear perturbation equations, but the scope of these solutions is limited. The specific problem of Couette flow has been treated by Velte [11][12] and Kirchgassner and Sorger[13]. The Bénard problem was discussed by Rabinowitz[14]. These solutions are extensions of the more general work on the spectrum of the eigenvalues of the Navier-Stokes equation by Prodi[15]. Sattinger[3] developed the more rigorous basis for the use of the "linearization hypothesis" for incompressible, viscous flows over closed domains.

The flows for which frictionless instabilities are important have been studied extensively but the number of solutions to various problems is very

limited. The work of Rayleigh as a starting point for these has been discussed earlier. There have been a number of other related works in divergent fields that directly impact on this problem. One was the work by Brunt[16] and Väisälä[17] concerning the stability of a density stratified atmosphere in a gravity field. Others in the field of meteorology include the appearance of Richardson and Rossby numbers as critical parameters to describe the effect of a density gradient, velocity gradient and rotation for stability in a gravity field. More recently the investigation concerning the design of swirl combustion chambers, flame holders and nozzles have added greatly to the knowledge of compressible swirling flows. In the latter, the experiments of Chigier[18] and Chervinski[19] increased the data base for the stability of flows with a superimposed swirl component.

The more recent analysis of Batchelor and Gill[20] in describing the incompressible jet have added a great deal to the knowledge of the stability of a straight jet. They showed that a jet with a top hat profile satisfies the necessary condition for instability to axisymmetric disturbances. In this case the instability has been found explicitly in the form of ring-shaped vortices at the boundary of the core of a jet as shown by Wehrmann and Wille[21]. It is these large scale and efficient emitters of noise that may produce a substantial part of the radiated noise by a jet engine.

The interest in swirling flows has not decreased since the development of the basic theorems in the early papers as is demonstrated by the large number of papers published in a variety of technical journals. The topics of these concern the behavior of applications for which the addition of swirl proved to be successful. Among these are the swirl combustors, flame holders

and nozzles. For some applications the introduction of the swirling velocity component was primarily used to stabilize a marginally stable system, in others it was used to enhance mixing. An example of stabilization is the flame behind an axisymmetric flame holder. The axisymmetric flow without rotation repeatedly tears and sheds vortices that are associated with the steady flow base flow, but if a sufficient azimuthal velocity component is added the flame is stabilized and holds its shape.

The number of analytically derived steady state solutions of swirling flow problems is limited; therefore, the criteria used to determine the stability of an arbitrary swirling flow must be very general. Lord Rayleigh's condition for the stability of an inviscid rotating fluid meets this requirement. The interactions that take place in a compressible swirling jet are numerous. First, the straight jet is unstable if the axial velocity component has a point of inflection. This is known via the second stability theorem[22]. Because the flow also has an azimuthal velocity distribution, it will be stable if the angular momentum increases outward from the surface. For compressible flow, if the buoyancy tends to return the flow to its original equilibrium condition, it corresponds to a statically stabilizing distribution. As a result the buoyancy and rotation may stabilize an otherwise unstable velocity distribution as based on the inflection point theorum. The guage stability, the ratio of the rotational components and the axial velocity gradients, results in a measure of the two effects that govern the overall stability of the flow field. This ratio, known as the Richardson number was originally derived to estimate the stability of a wind moving over a surface in a stratified atmosphere in a gravitational field. The fact

that a number of the particular problems scale similarly regardless of whether the central force field is generated by rotation or by gravity, allows generalization of the stability results for gravitational field to parallel stabilities for rotational flows.

The comparison of a straight and a swirling jet was made by N. A. Chigier[18] (Figure 2). If we remember that the existence of large scale structures present in the vicinity of the exit was proven by Wehrmann and Wille[21], some obvious conclusions based on the experiments of Chigier are most interesting. One of which could lead to a reduction in sound generated. Schwartz[24] postulated that the effect of a sufficient amount of rotation within a thin layer near the jet boundary greatly reduces the overall noise measured at a station in the far field. This could be due to a stabilization of the axisymmetric instability mode. In a paper by Mollo-Christensen[25], it was suggested that the near and far field pressure measurements could be correlated by assuming the sources of the sound were regularly distributed. He noted that a coherent large scale emitter would be much more efficient than randomly distributed emitters throughout the jet. In a later experiment Crow and Champagne[26] were able to excite different modes of the shear layer at the jet edge by imposing an acoustic source of a given frequency on a jet at various Reynolds numbers then observing both the behavior of the shear layer along with the emitted sound as a function of the downstream distance.

Brown and Roshko[27] observed the growth of structures in a two-dimensional shear layer and noted that such structures tend to grow with the linear distance from the shear layer origin. The modes of instability of

the jet would therefore play an important role in better understanding of the modeling of the turbulence structure within the jet itself. Michalke and Fuchs[28] used the cross correlation of particular modes to determine that jet noise can be determined using only the lowest modal oscillation associated with the large scale structures. This method provided a remarkable match to the actual data.

Laufer[29] used the measurements of other people to determine the separation distance between neighboring large scale structures (vortices) and found them to be a linear function of the distance from the jet origin. The propagation and growth of these vortices, their eventual pairing and the azimuthal modes may directly influence the measured nose by an observer in the field. (Figure 3)

There are therefore at least two motivations for the current effort. The first is to determine if the basic steady state flow will remain stable to infinitesimal disturbances. The second is to obtain a basic understanding of the stability of the jet relating to the production of noise and its propagation. It could be assumed that if the instability of the shear layer and the mixing process of the jet and the free stream are the primary sources of noise, stabilizing the shear layer by the introduction of an externally induced rotation could prove very beneficial. The earlier stability analysis for straight jets all indicate that the instability occurs where the velocity profile has a point of inflection. Near the jet exit this is in a thin layer, at the jet boundary therefore, the amount of swirl added to stabilize the flow can be small. This is important because replacing a potential flow field by one that has rotation will always result in lowered velocities and in the case of a jet lower thrust.

These motivations require an analysis that is concerned primarily with sources and modes of instability. Since for such perturbations the effect of viscosity is small, it is possible to analyze the perturbation from an inviscid point of view even though the steady state profiles were generated by viscous forces. The following analyses deal primarily with stability in time even though based on the compilation of Laufer a more meaningful investigation would also include a stability in space analysis. In chapter two this will be done for a flow field for which the steady state properties depend only on the radial distance. The basic stability equation remains in either case but the difference in definition of the operators that determine the frequency and wave number lead to different results.

Of the many papers that deal with the stability of rotating flows only a few were chosen to represent the ideas behind the development of the results shown here. One of the earliest was a paper by Howard and Gupta[30] that put limits on the rate of growth and propagation speed of the disturbances. This work extended the theorem that states that the velocity of the disturbance is bounded between the maximum and minimum velocities observed in the flow field. Leibovich[31] extended the method used by Howard and Gupta to problems that included axisymmetric disturbances in a density stratified medium. The results were the same except that the definition of the Richardson for rotational flows became the critical parameter. Fung and Kurzweg[32] extended the Leibovich results to include non-axisymmetric disturbances and discovered that the ratio of the wave number $S = \frac{\ell}{r\alpha}$ must be included to determine stability. In an earlier paper using the narrow gap

approximation for Couette flow, Yih[33] was able to show that for compressible flows the entropy variation normal to the stream surfaces must be known to obtain a simple criterion.

Because of the presence of vortex rings in the shear layer, the growth and decay of these is an important feature of the flow. Data that deals with the breakup of these rings is almost non-existant. Several papers that treat their growth and decay were written by Magarvey and MacLatchy[34][35]. These show that aged rings tend to break up due to the growth of azimuthal disturbances on the ring distorting the entire ring structure. In a gravitational field the breakup of the rings is associated with movement to positions of less gravitational potential. For a jet this would be accomplished by parts of rings moving to regions of smaller axial velocity gradients. Since the decay of the rotational velocity is approximately four times as rapid as the decay of the axial velocity component, the growth and destruction of vortices would correspond to regions axially removed from the exit. Rotation does inhibit the growth of ring shaped vortical structures, thus it is required to examine the experiments of Ludwieg[36][37] to determine the instability mechanism that is expected. Ludwieg provided an inviscid analysis to show that the disturbances that are associated with axially-symmetric Taylor vortices when the flow is in pure swirl become spiral instabilities when the flow is itself a spiral. This gives the spiral flow a central place in hydrodynamic theory along with axisymmetric straight and rotating Couette flow. Ludwieg's work was extended by Pedley[38] who showed that Ludwieg's results do not depend on the particular profile chosen. These experiments and theory may point to the proposition that the

axisymmetric disturbances are not always the most unstable and the intro-
duction of a swirl component will not always mean a more stable overall
system. In a paper dealing with the global stability of spiral flows,
Joseph and Munson[39] developed a linearized theory that was able to obtain
some results for the small gap approximation.

There is little hope that the above type of an analyses can be exten-
ded much beyond the results of chapter two. These would limit the appli-
cability of stability theory to a small number of problems actually observed.
It is possible under certain conditions to formulate the problem in Lag-
rangian coordinates by letting the characteristics of the steady flow be
functions of the position only. An equilibrium trajectory through space
would then be a description of a steady state flow. Perturbations would
then relate to the relative displacement of a fluid particle from the equi-
librium trajectory. Thus, stability would be determined by the condition
under which the displacement would decay to zero in time. This type of an
approach was developed from plasma physics where the importance of the vel-
ocity field would be of the first or second order. Several papers [41][42]
deal with this type of stability. The method called the "configurational
stability" approach will be followed in chapter three with the basic theory
derived for viscous effects in the direction of the velocity vector. Such
a system has the advantage of being in conservation form and allows looking
at the behavior of a particle of constant mass to see how its motion and
energy are affected by its interaction with the steady state field. The
resulting energy integral is taken over all the particles on a stream sur-
face to determine if it will decay in time.

The present analysis is twofold. In the first example the condition
of the swirling flow described by Lallas is extended to include the swir-
ling jet. This problem is more difficult first because the domain is
unbounded; and therefore, the energy integral must be redefined as was shown
by Carmi[48]. This involves not only defining the rate of decay of the
perturbation function as $r \to \infty$, but also verifying that the integral does
not have a singularity at $r = 0$. The boundary conditions must be redefined
so that the continuity equation is satisfied at $r = 0$. This analysis while
being fully compressible, inviscid and adiabatic in the perturbation quan-
tities still is restricted by the condition that the steady state depends
only on r. The second part of the analysis is used to find the condition
for which a stability condition can be determined for steady flows which
do not depend on the single coordinate but may vary in space. For this
purpose the mathematical formulation of Burshtein and Solov'ev[42] was
extended for compressible flow. These equations are cast in a Lagrangian
representation and all steady state variables are defined relative to the
equilibrium trajectory via a displacement vector $\vec{\xi}$. For the present example
it is assumed that $\vec{\xi}$ is normal to the stream surface at the point of interest.
The coordinate system chosen was in the shear surface and the normal to it.
This greatly simplified the equation of stability and allowed using an
integral method to obtain the condition that the imaginary part of $\vec{\xi}$ is
zero. This completed the stability analysis.

CHAPTER 1. BASIC EQUATIONS AND ASSUMPTIONS

In the introduction several sources were quoted that implied that the production of noise and the appearance of large scale vortical structures in a flow field are related. By inhibiting or reducing growth of these disturbances, it is hoped that a favorable reduction in the noise measured in the far field could be achieved. Even though there are multiple sources with different strengths and frequencies Michalke and Fuchs[28] showed that a substantial part of the overall noise spectrum and magnitude could be associated with the first few modes of the expected disturbances. This leads immediately to the appropriateness of a stability argument. If it is possible to determine the modes of instability and the associated flow field causes, the flow field may be tailored to give the most favorable over-all result.

Before any meaningful analysis can be made, certain basic assumptions regarding the flow field must be made. The most basic fluid dynamic equations weigh all the flow properties equally and thus produce a horrendous problem if all are retained in the stability analysis. This paper will deal with problems in which the mean or unperturbed flow depends only on space. The perturbed or fluctuating field may depend on both space and time. It will be assumed that the perturbations themselves are infinitesimal. This simplification eliminates the need to consider the magnitude of a disturbance when writing the equations of motion. It is also assumed that even though the derivatives and the perturbed quantities are not necessarily small, they will

always remain finite and will never be of sufficient size to alter the linearity assumption. This was discussed in the introduction concerning the basic assumptions regarding linearized stability theory. In general a disturbance may exhibit a number of properties that were described by Chandrasekhar[52], as unstable, marginally stable damped or marginally stable undamped. For linearized non-dissipative systems oscillations that are imposed on a conservative system either grow exponentially in time or are neutrally stable. If the growth in magnitude of the system is included, the oscillations can be limited in magnitude by the non-linear interactions that become important when the perturbations reach finite size. For dissipative systems the growth of a disturbance may be damped by the viscous forces, but for certain phasing of the viscous and thermal diffusion, the disturbance may be driven unstable (See Yih[33]).

Further assumptions must be made to make the perturbation system reduceable. At this point it is necessary to distinguish between the two methods employed. For the first or standard perturbation method the assumptions are: 1) Constant edge conditions or edge conditions that may be made constant via an integral transformation, 2) Constant momentum surplus. This is similar to the above except that it also eliminates sources of momentum or energy in the flow, 3) High Reynolds number. This assumption makes the free surfaces nearly straight so that assumption one is physically plausible for jet flows, 4) Ideal gas with no chemical reactions. Additional assumptions relate to the type of flow. The flow is assumed to be non-dissipative, fully compressible, with a basic flow that does not depend upon time. Each flow variable may be written as the sum of a mean steady state component plus

a fluctuating component with zero mean and separable time dependence. Thus:

$$q(\vec{x},t) = \langle q(\vec{x}) \rangle + q'(\vec{x},t) \qquad (1\text{-}1)$$

where q' may be made independent of time using an integral transform or an assumed form for the time dependence. To limit the complexity of the flow considered, the flow will be assumed to depend only on the radial direction. The problem will be solved in the entire plane using an integral approach with boundary conditions that give finite perturbation energy integral over the entire space. Because the integration takes place over all space the solution gives the sufficient condition for global stability. Since the mean flow depends only on r, the perturbation form in the remaining space variables is arbitrary. These will be assumed to be complex harmonics with constant wave numbers.

The higher order terms as well as the viscous terms are dropped for simplicity. The linearization assumption allows the first, and the second is possible because the viscous and inviscid solutions have been shown to be similar in the high Reynolds number limit. It has been shown by other authors that the effect of viscosity on the perturbation is small except in the vicinity of the critical layer. In the inviscid problem there exists a singularity at this layer. The effect of the viscosity is to reduce the gradient observed there over a finite region and to reduce their magnitude from infinity to a finite value. In examining the limit of instability (going from unstable to stable direction), the results should be very similar in either case. For damped oscillations the effect of the viscosity plays a very important role in determing the neutral stability curve as was shown by Tollmein[8].

The second example uses a Lagrangian approach to the stability problem. As before the properties of the flow field depend only on the space coordinates. The fate of a fluid particle of constant mass will be considered subject to the forces that act on it as it is displaced from its equilibrium trajectory. The entire flow field may thus be replaced by a variable force field that depends on the relative displacement from the equilibrium trajectory. The particle is identified by the trajectory that it is on at time t_o. Since the mass remains constant, the effect of compressibility is to change the volume of the fluid. The shape of the fluid volume made up of identifiable fluid elements is arbitrary; therefore, it is possible to define a coordinate system that describes its shape at time t on the initial trajectory and also its shape on the displaced trajectory via a curvilinear coordinate transformation. It is thus possible to define the effect of a laminar shear force in the perturbed and unperturbed examples. The result is that the entire stability problem is cast into a form that depends only on the relative displacement of the fluid particle.

This Lagrangian representation of stability is, therefore, a much more tenable conceptual approach to difficult stability problems because the number of unknowns is immediately reduced from seven to three. The inviscid equation that is produced is self-adjoint. This results in being able to use a whole class of mathematics developed for self-adjoint operators to determine the stability criteria. The viscous equations are not self-adjoint; and therefore, the results are not so easily obtained. The real part of the energy must be examined to determine the stability criteria. This method has been developed for use in plasma physics where the use of the inviscid

equations and or small relative fluid velocities are applicable. This is not
necessarily the case in fluid mechanics. For this reason the fully viscous
compressible stability equation was developed in chapter three. The assump-
tion of isentropic flow that was carried for all the previous inviscid methods
had to be dropped. Hence, the energy equation was altered substantially.
Two examples were carried because they result in substantially the same flow.
The first is to assume that the energy integral is a constant throughout
the flow and the second is to replace the isentropic relationship between
the pressure and density with one that depends locally on the entropy produc-
tion. The third alternative of using the entropy production form of the
second law of thermodynamics could also have been employed but it is left
for another analysis not to be presented here.

The starting equations of motion that will be used throughout are based
on the laminar form of the Navier-Stokes equations. These are given as
follows:

$$\frac{\partial \rho}{\partial t} + div\, \rho \vec{v} = \frac{\partial \rho}{\partial t} + div\, \vec{m} = 0 \tag{1-2}$$

$$\frac{\partial \rho \vec{v}}{\partial t} + div\,(\rho \vec{v}\vec{v} + P\underline{I}) = \frac{\partial \vec{m}}{\partial t} + div\left(\frac{\vec{m}\vec{m}}{\rho} + P\underline{I}\right) = 0 \tag{1-3}$$

$$\frac{DP}{Dt} = c^2\, \frac{D\rho}{Dt} \tag{1-4}$$

The perturbation equations are written by using the assumption of
equation (1-1) and retaining only the linear terms in the perturbation
quantities. The equation (1-2) and (1-3) are then written:

$$\frac{\partial \rho'}{\partial t} + \operatorname{div} \vec{m}' = 0 \tag{1-2a}$$

$$\frac{\partial \vec{m}'}{\partial t} + \operatorname{div}\left[\vec{m}'\frac{\vec{M}}{\rho} + \vec{M}\vec{m}' - \frac{\vec{M}\vec{M}}{\rho^2}\rho' + P'\underset{\sim}{I} \right] = 0 \tag{1-3a}$$

The divergence of the dyadic product may be expanded:

$$\operatorname{div}(\vec{M}\vec{M}) = \vec{M}\cdot\operatorname{grad}\vec{M} + \vec{M}\operatorname{div}\vec{M} \tag{1-5}$$

Differentiating equation (1-3a) with respect to time gives:

$$\frac{\partial^2 \vec{m}'}{\partial t^2} + \operatorname{div}\left[\frac{\partial \vec{m}'}{\partial t}\frac{\vec{M}}{\rho} + \frac{\vec{M}}{\rho}\frac{\partial \vec{m}'}{\partial t} - \frac{\vec{M}\vec{M}}{\rho^2}\frac{\partial \rho'}{\partial t} + \frac{\partial P}{\partial t}\underset{\sim}{I} \right] = 0 \tag{1-6}$$

With the continuity equation this becomes:

$$\frac{\partial^2 \vec{m}'}{\partial t^2} + \operatorname{div}\left[\frac{\partial \vec{m}'}{\partial t}\frac{\vec{M}}{\rho} + \frac{\vec{M}}{\rho}\frac{\partial \vec{m}'}{\partial t} + \frac{\vec{M}\vec{M}}{\rho^2}\operatorname{div}\vec{m} + \frac{\partial P'}{\partial t}\underset{\sim}{I} \right] = 0 \tag{1-6a}$$

Expanding the above equation:

$$\frac{\partial^2 \vec{m}'}{\partial t^2} + \frac{\partial \vec{m}'}{\partial t}\cdot\operatorname{grad}\frac{\vec{M}}{\rho} + \frac{\vec{M}}{\rho}\operatorname{div}\frac{\partial \vec{m}'}{\partial t} + \frac{\vec{M}}{\rho}\cdot\operatorname{grad}\frac{\partial \vec{m}'}{\partial t} + \frac{\partial \vec{m}}{\partial t}\operatorname{div}\frac{\vec{M}}{\rho} \tag{1-7}$$

$$\frac{\vec{M}}{\rho}\cdot\operatorname{grad}\left(\frac{\vec{M}}{\rho}\operatorname{div}\vec{m}\right) + \frac{\vec{M}}{\rho}\operatorname{div}\vec{m}\operatorname{div}\frac{\vec{M}}{\rho} + \frac{\partial}{\partial t}\operatorname{grad}P' = 0$$

The above equation may be further expanded to give the equation:

$$\frac{\partial^2 \vec{m}'}{\partial t^2} + \operatorname{grad}\left[\vec{V}\cdot\vec{A} + \frac{\partial P'}{\partial t}\right] - \vec{A}\times\vec{\Omega} - \vec{V}\times\operatorname{curl}\vec{A} + \vec{V}\operatorname{div}\vec{A} \tag{1-8}$$
$$- \vec{A}\left(\frac{\vec{V}}{\rho}\cdot\operatorname{grad}\rho\right) = 0$$

Where $\Omega = \operatorname{curl}\vec{V}$ and $A = \frac{1}{2}\vec{V}\operatorname{div}\vec{m} + \frac{\partial \vec{m}}{\partial t}$. Also from the steady state equation $\operatorname{div}\vec{M} = 0$ was used to simplify equation (1-8). Taking the curl of equation (1-8) results in:

$$\operatorname{curl}\left\{ \frac{\partial^2 \vec{m}'}{\partial t^2} - \vec{A}\times\vec{\Omega} - \vec{V}\times\operatorname{curl}\vec{A} + \vec{V}\operatorname{div}\vec{A} - \vec{A}\left(\frac{\vec{V}}{\rho}\cdot\operatorname{grad}\rho\right) \right\} = 0 \tag{1-9}$$

Let $\vec{m} = e^{iwt}f(\vec{r})$ where w may be complex, then $A = \frac{1}{2}\vec{V}\operatorname{div}\vec{m} + iw\vec{m}$

$$\operatorname{curl}\left\{ -\omega^2\vec{f} - \left(\frac{1}{2}\vec{V}\operatorname{div}\vec{f} + i\omega\vec{f}\right)\times\vec{\Omega} - \vec{V}\times\operatorname{curl}\left(\frac{1}{2}\vec{V}\operatorname{div}\vec{f} + \right.\right. \tag{1-10}$$
$$\left.\left. i\omega\vec{f}\right) - \vec{V}\operatorname{div}\left(\frac{1}{2}\vec{V}\operatorname{div}\vec{f} + i\omega\vec{f}\right) - \left(\frac{1}{2}\vec{V}\operatorname{div}\vec{f} + i\omega\vec{f}\right)(\vec{V}\cdot\operatorname{grad}\rho/\rho) \right\} = 0$$

Combine the above coefficients of w to determine the basic stability equation:

$$\omega^2 \text{curl}\,\vec{f} + \iota\omega \left\{ \text{curl}(\vec{f}\times\vec{\Omega}) + \text{curl}(\vec{v}\times\text{curl}\,\vec{f}) - \text{curl}(\vec{v}\,\text{div}\,\vec{f}) \right. \quad (1\text{-}10a)$$

$$+ \text{curl}\left[\vec{f}(\vec{v}\cdot\text{grad}\,\rho/\rho)\right]\bigg\} + \tfrac{1}{2}\,\text{curl}\left\{ \vec{v}\times\vec{\Omega}\,\text{div}\,\vec{f} + \vec{v}\,\text{curl}(\vec{v}\,\text{div}\,\vec{f}) - \right.$$

$$\vec{v}\,\text{div}(\vec{v}\,\text{div}\,\vec{f}) + \vec{v}\,\text{div}\,\vec{f}\,(\vec{V}/\rho\cdot\text{grad}\,\rho) \bigg\} = 0 \qquad (1\text{-}10a)$$

The less complicated vector equation in Lm is obtained if some assumptions are made regarding equation (1-4). Differentiate this equation with respect to time to obtain:

$$\frac{D}{Dt}\,\frac{\partial P'}{\partial t} = c^2\,\frac{D}{Dt}\,\frac{\partial \rho'}{\partial t} \qquad (1\text{-}11)$$

From the continuity equation it is possible to eliminate ρ':

$$\frac{D}{Dt}\,\frac{\partial P'}{\partial t} = -c^2\,\frac{D}{Dt}\,\text{div}\,\vec{m}' \qquad (1\text{-}12)$$

Find the total integral of equation (1-12):

$$\frac{\partial P'}{\partial t} = -\int c^2\,\frac{D}{Dt}\,\text{div}\,\vec{m}\,dt \qquad (1\text{-}13)$$

If $\vec{V}\cdot(\Delta P' - a^2\Delta\rho') = 0$, then:

$$\frac{\partial P'}{\partial t} = -c^2\,\text{div}\,\vec{m}' \qquad (1\text{-}14)$$

With this assumption equation (1-8) becomes:

$$\frac{\partial^2 \vec{m}'}{\partial t^2} + \text{grad}\left[\vec{V}\cdot\vec{A} + c^2\text{div}\,\vec{m}\right] - \vec{A}\times\vec{\Omega} - \vec{V}\times\text{curl}\,\vec{A} \qquad (1\text{-}15)$$

$$+ \vec{V}\,\text{div}\,\vec{A} - A\left(\vec{V}/\rho\cdot\text{grad}\,\rho\right) = 0$$

Again substituting for $\vec{m}$ yields:

$$\omega^2\vec{f} + \iota\omega\left\{ \vec{f}\times\vec{\Omega} + \vec{v}\times\text{curl}\,\vec{f} - \vec{v}\,\text{div}\,\vec{f} + \vec{f}\left(\tfrac{\vec{v}}{\rho}\cdot\text{grad}\,\rho\right) - \right. \quad (1\text{-}16)$$

$$\text{grad}(\vec{v}\cdot\vec{f}) \bigg\} + \tfrac{1}{2}\left\{ \vec{v}\times\vec{\Omega}\,\text{div}\,\vec{f} + \vec{v}\times\text{curl}(\vec{v}\,\text{div}\,\vec{f}) - \vec{v}\,\text{div}(\vec{v}\,\text{div}\,\vec{f}) + \right.$$

$$\vec{v}\,\text{div}\,\vec{f}\left(\tfrac{\vec{v}}{\rho}\cdot\text{grad}\,\vec{\rho}\right) + \text{grad}(\vec{v}\cdot\vec{v}\,\text{div}\,\vec{f} + 2c^2\text{div}\,\vec{f}) \bigg\} = 0$$

Either equation (1-10) or (1-16) may be used to evaluate a stability criterion subject to the appropriate boundary and initial conditions. The difference between the two is that (1-10) is the exact linear equation for inviscid, non-heat conducting, compressible fluids; whereas, equation (1-14) says something about the type of perturbations that will be admitted. If Doaks analogy of fluctuators that are either rotational or irrotational are assumed then further simplification of the equations are possible. Equation (1-10) for irrotational disturbances loses the leading term and is written as follows:

$$\omega \left\{ \operatorname{curl}(\vec{f} \times \vec{\Omega}) - \operatorname{curl}(\vec{v}\operatorname{div}\vec{f}) + \vec{f} \times \operatorname{grad}(\vec{v} \operatorname{grad}\rho) \right\} + \frac{1}{2}\operatorname{curl} \left\{ \ \right\} = 0 \quad (1\text{-}17)$$

Similarly for rotational perturbations the equation becomes:

$$\omega^2 \operatorname{curl}\vec{f} + \omega \left\{ \operatorname{curl}(\vec{f} \times \vec{\Omega}) + \operatorname{curl}(\vec{v} \times \operatorname{curl}\vec{f}) + \right. \quad (1\text{-}18)$$
$$\left. \operatorname{curl}\left[\vec{f}\left(\frac{\vec{v}}{\rho} \operatorname{grad}\rho\right)\right] \right\} = 0$$

Equation (1-16) is similar to (1-18) for rotational disturbance, but does not lose the highest order term for irrotational flow.

The equations are somewhat different from those used by other authors because the full compressible, inviscid equations have been used, and the main dependent variables are the mass flux, the pressure, and the density. Also the problems solved earlier dealt primarily with rotating Couette flow. The boundary conditions are of course different because the perturbation quantities must go to zero as $r \to \infty$. Since the boundary of the region is not changing, the perturbation coefficients must approach zero in a pre-scribed manner in order for the conservation equation to be satisfied. The initial conditions are also to be specified in order to have a well-posed problem.

The above equations are put into linear perturbation form with the assumption that all perturbation quantities have the form:

$$q = g(r)\, e^{i(\omega t + \alpha x + l\theta)}$$

The boundary conditions are rather complex. Since the surface of the jet is not assumed to fluctuate as a function of time, the velocity component and the fluctuating quantities must approach zero as $r \to \infty$ in a prescribed manner.

The velocity fluctuations have conditions as follows:

$$u',\, v',\, w',\, P',\, \rho',\, T' \to 0 \quad as \quad r \to \infty$$

Further for $l \neq 0$ $P = u' = 0$ at $r = 0$

For $l = 1$ ρ', v', w', may be finite at $r = 0$

     $l \neq 1$ v', $w' = 0$ at $r = 0$

No condition on ρ' and T' can be made at $r \to 0$ except the continuity and energy must be satisfied. The initial conditions are that at time $t = 0$ the perturbations are finite.

The statement of the problem using the Lagrangian approach will be left to Chapter three where it will be discussed in its entirety.

CHAPTER 2. STABILITY OF STEADY FLOWS WITH RADIUS DEPENDENT PROPERTIES

In the previous chapter we outlined the works of authors that con-
cerned themselves primarily with axisymmetric flows. The earliest papers
dealt exclusively with incompressible media such as water; and therefore,
the equations that were used to determine the stability limits were rather
simple. There are several particular flows whose stability has been examined
for a variety of conditions. These are Couette flow between two concentric
rotating cylinders, pipe or Poisuelle flow and the axisymmetric jet. Each
of these flows had been examined experimentally much earlier than the first
mathematical theories that could give some insight on the behavior of the
steady state subjected to a variety of perturbations and boundary condi-
tions[1][2]. As stated previously, Lord Rayleigh was the first to describe
the behavior of the jet and a rotating flow field and proposed several
theorems that govern the behavior of the stability of a variety of more
general fluid dynamic motions. These were stated concisely as: 1) An
inviscid rotating flow is unstable if the square of its circulation de-
creases outward, 2) The flow with a velocity disturbation such that it
has a point of inflection insures instability. The first is known as Ray-
leigh's criterion for rotating flow[4]. Karman explained the first law
by stating that a radially displaced particle would encounter an insuffi-
cient pressure gradient to return it to its original condition. Adding
viscosity to the fluid would add retarding force against the particle motion

and thus weaken the criterion. For the second theorem Rayleigh was only
able to show that the condition of a point of inflection was a necessary
condition for instability. It was not until much later that Tollmien[8]
was able to show that this was also a sufficient condition for the ampli-
fication of disturbances. The inflection point criteria could be related
to the pressure and its effect on the behavior of the flow. The influence
of viscosity on the above discussion changes the conclusions only slightly;
therefore, these instabilities may be regarded as "frictionless instabili-
ties" because the flow proves to be unstable even without accounting for
the effect of viscosity. (Fig. 3)

By considering only amplified or neutral disturbances, the effect of
viscosity is small in that the disturbances satisfy the inviscid equation
of motion. This is seen by assuming that $1/R$ is very large in the viscous
Orr-Sommerfeld equation. The effect on these disturbances is small provided
that when the singularity appears at the critical point, the same branch
of the logarithm is retained on either side of the singularity. Lin[9]
has shown that for damped oscillation, the path of integration switches
to the other side of the real axis at the critical point. Thus only ampli-
fied and neutral disturbances can be found by a straight-forward integration
of the stability equation. In either case the viscous layer is thin and
thus the effect of viscosity in the limit on the axisymmetric flow is the
same as for a two-dimensional flow.

Instability of a given flow usually implies that several types of
changes take place after the steady flow has been subjected to an arbitrary
disturbance of undetermined amplitude and wave form. For any system there

is always a disturbance amplitude, regardless of the restoring forces present, that will completely alter the flow field character. This corresponds to a particle in a potential well that has just been given enough kinetic energy to escape from the well. Instability may also result from the interaction of a steady flow and the perturbation. In this example, the initial amplitude of the disturbance may be important but its wave shape also has significance and may determine whether the disturbance will grow or decay. This is typically the phenomenon that is postulated when instability results after imposing an infinitesimal disturbance. A flow may be locally stable as described by a particle on a wavy surface going from one stable state to another but not necessarily returning to its initial state. Each low point on the surface corresponds to a local minimum of the potential energy. Thus, as far as the particle is concerned, stability is determined by a minimum potential energy. For fluid dynamic aspects the same situation holds except that the definition of the potential and kinetic energies is much more complex than for a single particle on a surface. Examples that correspond to each form of instability are (1) a dam bursting, (2) a flow becoming turbulent after a finite transition length and (3) a flow field is still laminar but a new solution has bifurcated that satisfies all the boundary conditions of the original solution. This was the problem that was considered by Kirchgassner and Sorger[13] for Couette flow.

In this chapter we will deal only with a flow that has radius dependent steady state variables. This is the problem that has been solved in various forms by virtually all authors who have considered a stability analysis. This assumption reduces the problem to either a one-dimensional

or a quasi one-dimensional steady state flow as was done by Lin[9] and Sommerfeld[56]. The differential equation of stability is in the form of an eigenvalue problem that can be solved for the eigenvalues to determine the stability limits.

The next more difficult problem consisted in the superposition of rotating Couette and Poisuelle flows for which the steady state velocity components were functions only of the radial coordinates. Most of the above examples involved flows with finite domains, but it was extended to infinite domains for incompressible flow by Carmi[48]. Density stratified flow was also studied by Leibovich[31] resulting in a Richardson number criteria that was similar to the one obtained by Yih[33] in the approximation of a viscous fluid in a thin layer between two rotating concentric cylinders.

By assuming instability Lallas[44] was able to derive a condition for compressible flow that contained the Brunt-Väisälä frequency in an integral manner. The Richardson number and Brunt-Väisälä[17] frequency are parameters that were derived by meteorologists studying the stability of the earth's atmosphere and the behavior of large air masses. Both apply to density variable flows in the presence of a central force field. Since they appear in other compressible and rotational flows, the universal nature of the underlying causes for the instability must be involved in the creation of a buoyancy force.

We will now consider a radius dependent, non-heat conducting inviscid flow sufficiently far removed from its source so that it is established and axial changes in the steady flow properties over one wavelength of the disturbance are negligible. For an incompressible jet, the conditions

were fully discussed by Batchelor and Gill[20] and the reader is asked to read their paper for a further discussion. The present will be an extension of their and others work to a fully compressible flow.

We will start by assuming that the primary flow conditions are steady and hydrodynamic equation of motion are satisfied. The steady flow in addition is assumed to be nearly parallel so that the x-dependence is unimportant.

As in chapter one, the steady state variables are perturbed to obtain the system of equation that needs to be solved to obtain the condition for the stability of the stationary hydrodynamic system. The governing equations are linearized in the perturbation variables, with an arbitrary perturbation variable in the form: $q' = q(r)e^{i(\omega t + \alpha x + \ell \emptyset)}$. Stability in time is desired; therefore, conditions are determined for which ω is a real quantity.

For a jet two forms of instability are very apparent. The first is observed near the exit of a jet and is the axisymmetric ring vortex that forms when there is an inflection in the steady velocity. These were observed by Wille and Wehrmann[21] for a liquid into liquid jet. The next is an instability of a spiral nature that was examined by Ludwieg[36]. The global nature of the instability demonstrated by Ludwieg was studied mathematically for incompressible flow by Joseph and Munson[39].

The starting equations for the perturbed variables are assumed to be the non-heat conducting, time dependent Euler equations. The energy equation used has no dissipation terms which implies that the entropy and its perturbations are constant along streamlines. The coordinate system chosen is

cylindrical:

Continuity:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v +\frac{1}{r}\frac{\partial}{\partial \theta}(\rho w +\frac{\partial}{\partial z}(\rho u)=0 \tag{2-1}$$

x-momentum:

$$\rho\left(\frac{\partial u}{\partial t}+v_r\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}+u\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial z} \tag{2-2}$$

r-momentum:

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \theta}-\frac{w^2}{r}+u\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial r} \tag{2-3}$$

θ-momentum:

$$\rho\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \theta}+\frac{vw}{r}+u\frac{\partial w}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{2-4}$$

Energy:

$$\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial r}+\frac{w}{r}\frac{\partial P}{\partial \theta}=-\gamma P\left[\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial vr}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right] \tag{2-5}$$

The above equations are expanded in a Taylor series for the quantities
ρ, P, u, v, and w. Only the linear terms are retained in the above system
of equations. Because of the assumption that the flow is nearly cylindri-
cal the steady state velocity component v is much smaller than the remaining
velocity components. Therefore, it is possible to assume that $v \approx 0$ within
the scope of this problem. The steady state properties are a function of
the radial distance only; this allows replacing the perturbation functions
with the following form:

$$q'(r,\theta,x,t) = q(r) e^{i(\omega t + \alpha x + \ell\theta)}$$

The circumferential velocity may be written $w = r\Omega$. The derivatives
with respect to r will be denoted by the symbol $D = \frac{d}{dr}$, $Df = r\frac{d}{dr}\left(\frac{f}{r}\right)$
and $D^*f = \frac{1}{r}\frac{d}{dr}(rf)$. An operator that is useful in contracting the equa-
tions is $L = \omega + u\alpha + \ell\Omega$. Finally, it is possible to define a new variable
$\rho v' = m$, the normal mass flux perturbation. The equations of motion are

then written for the linearized perturbations.

$$iL\rho' + i\rho(\alpha u' + \ell\Omega') + D^* m = 0 \tag{2-6}$$

$$iLu' + \frac{m}{\rho}Du + i\alpha\pi/\rho = 0 \tag{2-7}$$

$$iLm' - \rho'\Omega^2 r - 2\rho\Omega\Omega' r + D\pi = 0 \tag{2-8}$$

$$iL\Omega' + m D^*(r\Omega)/r\rho + i\ell\pi/\rho r^2 = 0 \tag{2-9}$$

$$iL\pi + \frac{m}{\rho}DP = -\gamma P L i(\alpha u' + \ell\Omega') + D^* m/\rho = i a^2 L\rho' + a^2 D^* m \tag{2-10}$$

Now eliminate the perturbation quantities u' and Ω'. Then the three equations are given by:

$$iL^2\rho' - (\alpha^2 + \ell^2/r^2)\pi - (\alpha Du + \ell D^*(r\Omega)/r)m + L D^* m = 0 \tag{2-11}$$

$$-L^2 m - iL\rho'\Omega^2 r + 2\Omega D^*(r\Omega)m + 2\Omega\ell i\pi/r + iLD\pi = 0 \tag{2-12}$$

$$iL\pi - i a^2 L\rho' = -\frac{m}{\rho}DP + mD\ln\rho \tag{2-13}$$

Again it is possible to simplify the above equations by defining a new operator $B = \alpha Du + \ell D^*(r\Omega)/r = DL + \frac{2\Omega\ell i}{r}$. The angular momentum gradient is given by $\Omega_R = 2\Omega D^*(r\Omega) = D(r^2\Omega)^2/r^3$ and the angular Mach number gradient or the normal pressure gradient is given by $M = \Omega^2 r/a^2$. The wave number of the disturbance is given by $e = \alpha^2 + \ell^2/r^2$. It is now possible

to eliminate the density perturbation ρ' from equations (2-11) and (2-13).

$$iL\rho' = ie\pi/L + Bm/L - D^+ m$$

After substitution for $iL\rho'$, equation (2-12) becomes:

$$L^2 m - \Omega^2 r \left(\frac{ie\pi}{L} + \frac{Bm}{L} - D^+ m \right) + \Omega_R m + 2\pi\Omega i/r + iL D\pi = 0$$

Equation (2-13) may be rewritten:

$$i \left(L^2/a^2 - e \right) \pi = Bm - LD^+ m + (Mm + mD\ln\rho)L$$

Combining terms gives:

$$iA\pi = L(D\ln\rho - M)m + mB - LD^+ m \qquad (2\text{-}13')$$

$$(\Omega_R \cdot L^2)m - m\Omega^2 rB/L + \Omega^2 r D^+ m = i\Omega^2 re\pi/L - 2\pi\Omega i/r \qquad (2\text{-}14)$$
$$+ iL D\pi$$

This reduces the number of equations and unknowns to two. The pressure perturbation, π, may be eliminated by substituting for π from equation (2-13') into equation (2-14).

$$\rho L D \left(\frac{D^+ m}{\rho A} \right) + mL \left(1 - \frac{\Omega_R}{L^2} - \frac{N^2}{L^2} \right) + \frac{ML - B}{A} (D\ln_A - M)m + \qquad (2\text{-}15)$$

$$M Bm/A - mD_*(B/A) - Lm D_* \left(\frac{D\ln\rho - M}{A} \right) - \frac{2\Omega i}{rAL} Bm = 0$$

In order to determine a sufficiency condition that is applicable to the most general situation that can be described based on the initial assumptions, a new variable is defined given by $L^{\frac{1}{2}}y = m$. Divide through by ρ_o and substitute into the above equation; then divide by the coefficient of $D\left(\frac{L}{\rho_o A} D^+ y \right)$. Now the equation becomes:

Let $C = \frac{DL}{2} + \frac{2\Omega i}{r}$

$$D\left(\frac{L}{\rho A}D^*\psi\right) - \frac{1}{\rho}\left\{-\frac{c^2}{AL} + L\left(1 - \frac{\Omega_R}{L^2} - \frac{N^2}{L^2}\right) + \frac{ML}{A}(D\ln\rho - M)\right.$$

$$\left.\frac{1}{A}(D\ln\rho - M)(c + DL) - D_*\left(\frac{c}{A}\right) - LD_*\left(\frac{D\ln\rho - M}{A}\right) + \frac{Mc}{A}\right\}\psi = 0 \qquad (2\text{-}16)$$

The mean flow is axisymmetric but the disturbances may be either axisymmetric or have a circumferential distribution. In the above equation are several terms that represent physical quantities that are present in flow for which there is a force field. The term $(M - D\ln\rho_0)$ is the numerator of the Vaisala-Brunt frequency. This frequency was first observed in the calculation of the stability of the stratified atmosphere. Vaisala showed that in order for the atmosphere to be stability stratified the balance of the gravitational force and the bouyancy force must be related in a particular manner. For adiabatic flow the relationship results in particular distribution of the temperature, density and pressure with altitude. This distribution results in a decay of the temperature known as the adiabatic lapse rate. The terms $\Omega_R + N^2$ and C^2 combine to form the Richardson number $(\Omega_R + N^2)/C^2$. This number relates the stability of a flow that has a velocity gradient in the direction of a force field. In this case the force field is generated by the centripetal acceleration.

The above equation differs from those of the authors that have assumed density stratified flow in that the speed-of-sound becomes an important parameter in discussing the behavior of the stability of the flow. Here, the operator A is a complex quantity, but for density stratified flow this operator reduces to the expression -e, a real number. As a result, the stability parameter must be more complicated than those derived for density

stratified flows. The most recent has been by Lallas[44]. Though he starts with the same basic equations, he uses several different transformations that reduce his dependent variable to a complicated expression of the velocity, pressure and speed-of-sound. In our case, the dependent variable is simply a function of the normal mass flux. Therefore, the expressions of stability must be somewhat more complicated than his. After a substantial amount of mathematical manipulation we derive the result:

$$-\frac{4c^2}{L^2} + \left[\frac{c^2}{|L|^2} + \left(M + \frac{2}{r}\right)\frac{DS}{c_p}\right] + 2L_R c\left[M + \frac{2}{r} + \frac{DS}{c_p}\right] + 4DL\frac{DS}{c_p} - |Aa|^2\left(1 + \frac{\Omega_R^2}{|L|^2} + \frac{N^2}{|L|^2}\right) > 0$$

The above does contain all the terms of Lallas plus a few extra ones. Stability is an intrinsic quality of the flow field and its limits should not be altered with a different choice of dependent or independent variables. Since the two analyses have the same initial starting point, the inviscid equation of motion and appropriate boundary conditions, the two results should be identical.

Let:
$$C = \frac{DL}{Z} + \frac{2\Omega\ell}{r}$$

Then equation (2-16) may be written in integral form:

$$-\int D\left(\frac{L}{\rho A}D^*\psi\right)\psi^* r\,dr + \int\frac{r\,dr}{\rho}\left\{-\frac{C^2}{AL} + L\left(1 - \frac{\Omega_R}{L^2} - \frac{N^2}{L^2}\right) + \frac{ML}{A}(D\ell n\rho - M)\right.$$
$$\left. - \frac{1}{A}(D\ell n\rho - M)(C + DL) - D_*\left(\frac{C}{A}\right) - LD_*\left(\frac{D\ell n\rho \cdot M}{A}\right) + \frac{MC}{A}\right\}|\psi|^2 = 0$$

The derivatives $D_*\left(\frac{C}{A}\right)$ and $LD_*\left(\frac{D\ell n\rho}{A}\right)$ are the ones that lead to difficulty. Since they are given as a derivative of a complex quantity, the first term may also be written as follows:

$$\int r|\Psi|^2 \left[D(C/A) - \frac{C}{Ar} \right] dr = -\int \frac{C}{A} |\Psi|^2 dr + \int r|\Psi|^2 D(C/A) dr$$

$$= -\int \frac{C}{A} |\Psi|^2 dr + \int r \Psi^* \Psi \, D(C/A) dr$$

The last integral may be integrated by parts to yield:

$$r|\Psi|^2 \frac{C}{A} \Big|_{r_0} - \int \frac{C}{A}\left[\Psi D(\Psi^*/r) + r\Psi^* D\Psi \right] dr = -\int \frac{C}{A}\left[|\Psi|^2 + r(\Psi D\Psi^* + \Psi^* D\Psi) \right] dr$$

Since the integral is to be taken from r = 0 to r → ∞, care must be taken so that the value of $r|\Psi|^2 \frac{C}{A} \to 0$ as r → ∞. This implies that regardless of the value of C/A as r → ∞, $|\Psi|^2$ must go to zero as 1/r. Therefore:

$$\int r|\Psi|^2 D_\alpha(\frac{C}{A}) dr = -2\int \frac{C}{A}|\Psi|^2 dr - \int \frac{C}{A} r(\Psi D\Psi^* + \Psi^* D\Psi) dr$$

The condition at r = 0 forces the value of Ψ to approach zero as:

$$\Psi \sim r^\alpha \qquad r\Psi D\Psi = r^\alpha r^{\alpha-1} r \qquad r\alpha = r^{2\alpha} \alpha \text{ implies } \alpha > 0$$

$$\text{If} \qquad C/A \to const \quad as \quad r \to 0$$

The second integral is written as:

$$-\int rL|\Psi|^2 D_r\left(\frac{D\ln\rho - M}{A} \right) dr = \int \frac{D\ln\rho - M}{A} D(rL|\Psi|^2) dr + \int |\Psi|^2 \frac{L}{A}($$

$$D\ln\rho - M) = \int \frac{D\ln\rho - M}{A}\left[2L|\Psi|^2 + r(DL|\Psi|^2) + rL(\Psi D\Psi^* + \Psi^* D\Psi) \right] dr$$

Equation 2-17 becomes:

$$-\int D(\frac{L}{A} D^*\Psi)\Psi^* r \, dr + \int \frac{r|\Psi|^2}{\rho}\Big\{ -\frac{C^2}{AL} + L(1 - \frac{Q_t}{L^2} - \frac{N}{L}) + \frac{ML}{A}(D\ln\rho -$$

$$M) - \frac{1}{A}(D\ln\rho - M)(C + DL) + \frac{2C}{A} + \frac{MC}{A} r \, 2(D\ln\rho - M)/Ar + (D\ln\rho - M)DL/A\Big] dr$$

$$+ \int r\left(\frac{C}{A} + L\frac{(D\ln\rho - M)}{A} \right)(\Psi D\Psi^* + r\Psi^* D\Psi) dr = 0$$

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Integrating the first integral by parts yields:

$$\int D\left(\frac{L}{\rho A} D^+ \psi\right) r \psi^* dr = \frac{L r \psi^*}{\rho A} D(\psi) - \int \frac{L}{\rho A} D^+ \psi \, D(\psi r) \, dr$$

where

$$D^+ = D + \frac{1}{r} \Rightarrow \frac{1}{r} \mathfrak{D}(r \psi)$$

The above integral becomes $-\int \frac{Lr}{\rho A} |D\psi|^2 \, div.$ Then the above integral

relationship becomes:

$$\int |D\psi|^2 \frac{rL}{\rho A} dr + \int r|\psi|^2 \left\{ -\frac{C}{AL}^2 + \frac{C}{A}\left(M + \frac{z}{r}\right) + L\left(1 + \frac{\Omega_z^2}{L^2} + \frac{N}{L^2}\right) \right.$$

$$\frac{L}{A}(D\ln\rho - M)(M + \frac{z}{r}) + \frac{1}{A}(D\ln\rho_0 - M)(C + 2DL)\Big] dr + \int r\left(\frac{C}{A} + \frac{L}{A}\right)(D\ln\rho - M)\Big[$$

$$\ldots \psi^* + D\psi\Big] dr = 0$$

There is a common factor given by $\frac{C}{A} + \frac{L}{A}(DL\rho_0 - M)$ in the above equation.

Find the imaginary part of the above expression to determine a condition

that will yield L_I as a common factor:

$$\frac{L}{A} = \frac{L_R}{a^2 |A|^2}\left[L_R^2 - L_I^2 - ea^2\right] - \frac{L_I}{a^2 |A|^2} r \left[L_R^2 + L_I^2 - ea^2\right]$$

$$\frac{1}{AL} = \frac{L_R}{a^2 |A|} r \left[L_R^2 - 3L_I^2 - ea^2\right] - \frac{i L_I}{a^2 |A| L_I^2}\left(3L_R^2 - L_I^2 - ea^2\right)$$

$$\frac{1}{A} = \frac{1}{|A|^2 a^2}\left(L_R^2 - L_I^2 - ea^2\right) - \frac{2 L_R L_I}{a^2 |A|^2} i$$

$$\frac{1}{L} = (L_R - i L_I)\frac{1}{|L|^2}$$

The imaginary part of the above integral becomes:

$$L_I \left\{ \int \frac{1}{a|A|^2} \left[L_R^2 + L_I^2 + ea^2\right] |D\psi|^2 r \, dr + \int r \frac{|\psi|^2}{a^2 |A|^2} \left\{ -\frac{L_R^2}{|L|^2} + \frac{C^2}{|A|^2}\left(L_R^2 + L_I^2 + ea^2\right) \right. \right.$$

$$+ (M + \frac{z}{r})\left[2 L_R C + (L_R^2 + L_I^2 + ea^2)(D\ln\rho - M)\right] - a|A|^2\left(1 + \frac{\Omega_z}{|L|^2} + \frac{N^2}{|L|^2}\right) + 2L_R(C + DL)$$

$$(D\ln\rho - M)\Big\} dr - \int \frac{r}{|aA|^2} \left\{ 2L_R C_1 (L_R^2 + L_I^2 + ea^2)(D\ln\rho - M)\right)\Big[\psi^* D\psi + \psi D^+ \psi^*\Big] dr\right\} = 0$$

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The last integral could prove troublesome because even though $\psi D\psi^* +$ $\psi^* D\psi$ is real, its sign and magnitude are not determined. By forming perfect squares with the term in the first integral it can be shown that the contribution of the last interval is less than that of a comparable term with $|\psi||D^*\psi|$ as its multiplier and a positive sign. Elimination of the last and first integrals simplifies the expression to one of proving that the value of the integral of the second term is greater than zero for all r to guarantee that L_I is not zero. With a little rearranging it can be shown that the condition to guarantee that the flow is stable is first to prove that only a value of $L_I^2 < 0$ can make the integrand zero and that the integrand of the second equation is positive, i.e.;

$$I_2 = \frac{c^2}{|L|^2} + \left(M + \frac{\gamma}{r} \right) \left(D \ln \rho - M \right) > 0$$

If the thermodynamic equations are examined it is possible to show that:

$$D \ln \rho - M = \frac{1}{c_p} \left(\frac{D\rho}{\rho} - \frac{DP}{\gamma P} \right) = \frac{Ds}{c_p}$$

which is the gradient of the entropy from streamline-to-streamline of the steady state or mean flow. The method used to generate the distribution is left to the imagination of the reader for a particular problem. The only assumption on the steady flow is that the effects of the shear do not affect the perturbation quantities:

$$- \frac{4c^2}{|L|^2} + \left[\frac{c^2}{|L|^2} + \left(M + \frac{2}{r} \right) \frac{Ds}{c_p} \right] + 2Lc \left[M + \frac{2}{r} + \frac{Ds}{c_p} \right] + 4 \frac{DL Ds}{c_p} + |\Lambda a|^2 \left(1 + \frac{\Omega r N^2}{|L|^2} \right)$$

The above criterion for stability differs in sign from the results of previous authors because the choice of the pressure operator A was chosen to have the opposite sign. In the incompressible limit the results compare

with those of Fung and Kurzweg. The fully compressible result is somewhat different from that of Lallas[44] because of the inclusion of terms that relate to the entropy gradient normal to the streamlines. Lallas used the normal velocity perturbation as his final parameter, whereas, the normal perturbations mass flux was used in this analysis. This recasts the equations to one in which the changes to a fluid element as it moves from streamline to streamline are included. The final stability criterion is then written in terms of the primary quantities. The effect of rotation is included directly through the Rayleigh discriminant Ω_R and through the Väisälä-Brunt frequency N^2. In the latter an acceleration field is produced by the rotation and as Miles[23] and Chimonas showed, this provides an effect that is similar to having a centralized force field such as gravity. The additional terms involve the velocity gradient, the square of the circulation M and the entropy DS/Cp.

$$(l_{\phi}^2 + c\alpha^2)\left[\frac{c^2}{|L|^2} + (M + \frac{2}{r})\frac{DS}{C_p}\right] + 2L\kappa c\left[M + \frac{2}{r} + \frac{DS}{C_p}\right] + 4DL\frac{DS}{C_p} - \frac{\Omega_r^2 + N^2}{|U|^2}|A\alpha|^2 \quad (2-18)$$
$$- \left[\frac{4c^2}{|L|^2} + |A c|^2\right] > 0$$

The above should be taken in the context that the definition of effects the sign. As stated before, the above condition for nonaxisymmetric disturbances reduces to the criteria of density stratified axisymmetric and non-axisymmetric flows and also reduces in the limit of incompressible non-rotating jets to the results of Batchelor and Gill[20] if the starting point of the analysis is equation (2-14) the final differential eigenvalue equation in the stability variable m. The expression that a point of inflection with respect to the steady state properties and the wave numbers of the flow are given by the term $D_*\left(\frac{B}{A}\right)$. For the incompressible flow this reduces to $D_*\left(\frac{\alpha Du}{\alpha^2 + \ell^2/r^2}\right)$, and this recovers the condition regarding points of

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inflection. One of the simplest problems should be that of two uniform flows separated by a contact discontinuity. In the two "outer" regions, the differential equation of motion is given by a reduction of equation (2-14) or (2-17) in the transformed variable. For no rotation this reduces to:

$$\frac{L}{\rho A} D(D\psi) - \frac{rL\psi}{\rho} = c \quad \text{or} \quad D(D^*\psi) + rA\psi = 0 \tag{2-19}$$

The above is a Bessels equation if the geometry cylindrical and an equation for exponentials if the surface is straight. In order to obtain the condition across the surface one may integrate or transform it directly:

$$\Delta_s \left[\frac{c}{\rho A} D^*\psi \right] + \lim_{\varepsilon \to 0} \int_{r-\varepsilon}^{r+\varepsilon} \psi \left[-\frac{c}{AL} - LD_*\left(\frac{Ds}{C\rho A}\right) - \frac{1}{A} \frac{Ds}{C\rho}(c+DL) \right] dr = 0$$

$$\Delta_s \left[\frac{c}{\rho A} D^*\psi \right] - (\psi L)_s \Delta_s\left(\frac{Ds}{C\rho A}\right) - \frac{\psi}{A}\Big|_s \left[\Delta \frac{Ds}{C\rho}(c+DL)_s + \frac{Ds}{C\rho}\Big|_s \Delta(c+DL)_s \right.$$

$$\left. - \frac{\psi}{AL}\Big|_s 2A(c\int c) \right] = 0$$

Because the flow is assumed constant on either side of the boundary, the derivatives must be equal to zero and the condition reduces to:

$$\Delta_s \left[\frac{c}{\rho A} D^*\psi \right] = c \tag{2-20}$$

where

$$\Delta_s f = f_{r+\varepsilon} - f_{r-\varepsilon}$$

The solution must go to zero as $r \to 0$ and as $r \to \infty$. For a cylindrical surface, this condition is not sufficient to eliminate a constant for the region $r > r_o$. Thus an additional constant such as the Sommerfeld radiation condition must be imposed to reduce the equation. Across the boundary the displacement of the surface must be the same. Thus, the second condition is that:

$$\Delta_3 \left(\Psi / L \right) = 0 \tag{2-21}$$

The outer solution is written in terms of the Hankel function or cylinder functions:

$$D^2 \Psi + \tfrac{1}{r} D\Psi + \left(L^2/a^2 - \alpha^2 + \frac{\ell^2+1}{r^2} \right) \Psi = 0$$

$$\Psi = A_1 J_\nu \left(\sqrt{L^2/a^2 - \alpha^2} \; r \right) \quad \text{where} \; \nu = \sqrt{\ell^2+1} \qquad \text{inner region}$$

$$\Psi = A_2 H_\nu^{(1)} \left(r \sqrt{\frac{L^2}{a^2} - \alpha^2} \right)$$

Condition (2-21) gives the condition:

$$L_{outer} \left(A_1 J_\nu \left(r_3 \sqrt{L^2/a^2 - \alpha^2} \right) \right) = L_{inner} \; A_2 H_\nu^{(1)} \left(r_3 \sqrt{L^2/a^2 - \alpha^2} \right)$$

or

$$A_2 = \frac{A_1 \; L_{outer}}{L_{inner}} \; \frac{J_\nu}{H_\nu^{(1)}}$$

And condition (2-20) gives:

$$\frac{L}{\rho A} \Big|_{inner} \left(D\Psi + \Psi/r_3 \right)_{inner} = \frac{L}{\rho A} \Big|_{outer} \left(D\Psi + \Psi/r_3 \right)_{outer}$$

$$D \left(r^\nu Z_\nu (r) \right) = \nu r^{\nu-1} Z_\nu (r) + r^\nu D Z_\nu (r) = r^\nu Z_{\nu-1} (r)$$

$$\frac{L^2}{\rho A} \Big|_{inner} H_\nu \left[J_{\nu-1} + \frac{-\nu}{r_3} J_\nu \right]_{inner} = \frac{L^2}{\rho A} \Big|_{outer} J_\nu \left[H_{\nu-1}^{(1)} + \frac{-\nu}{r_3} H_\nu^{(1)} \right]_{outer}$$

The above transendental equation is to be solved in such a manner such that w is real. Even for this very simple problem the introduction of the cylindrical geometry leads to a very complicated problem that cannot be solved for the nonaxisymmetric case $\nu \neq 1$. If $\ell = 0$, $\nu = 1$, and the above condition becomes:

$$\frac{L^2}{\rho A} \Big|_{inner} H_1^{(1)} J_o = \frac{L^2}{\rho A} \Big|_{outer} J_1 H_o^{(1)}$$

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Even in this very simplified statement a solution is possible only for the two dimensional limit. In chapter four the same problem will be repeated using a potential inner and outer flow. It will show that for the cylindrical geometry there are regions of stability and instability if there is no central force or acceleration field.

This example shows that as soon as more complicated shapes for the discontinuity are assumed the equations become most difficult to solve analytically. Therefore, in order to obtain any information regarding the stability of a flow field regardless of the frequency, wave number and the numerical form of the steady state variables, the results of a simplifying analysis is required.

Not all forms of stability are necessarily related to the growth of a disturbance in time. The more likely condition for some problems is that a disturbance appears at a point in the flow field and due to the interactions that occur between it and the steady flow the disturbance grows as it is convected with the fluid. If the steady state properties depend only on the radial distance from a centerline then the differential equation of stability retains the same form as equation (2-16). The difference is that the operators L and A have real and imaginary components of the axial wave number α rather than the frequency. For this problem stability in space may also be ascertained by using a difference energy approach as was done earlier in the chapter. The new definitions of the real and imaginary parts of the operators must now be determined to find the new criterion.

The procedure that is used follows directly from the above analysis whereupon we arrive at the integral relationship as before:

$$\int |D\psi|^2 \frac{r}{\rho A} dr + \int r |\psi|^2 \left\{ - \frac{c^2}{AL} + c(m + \frac{3}{r})/A + L(1 - \frac{2c}{L^2} - \frac{N^2}{L^2}) \right.$$

$$\left. + \frac{c}{A}(D\omega\rho - M)(M + \frac{3}{r}) + \frac{1}{A}(D\omega\rho - M)(c + DL) \right\} dr + \int r \left[\frac{c}{A} + \frac{1}{A}(\rho\omega\rho - M) \right]$$

$$\left[\psi^* D\psi + \psi D^* \psi^* \right] dr = 0$$

The above equation must be divided into real and imaginary parts. For stability in time $DL = DL_R$, but for stability in space $L_I = U\alpha_I$ and the derivative with respect to r is not only a real number. The additional terms that must be evaluated are then given as follows:

$$c = DL + \frac{2\Omega c}{r} = DL_R + i\alpha_I Du + \frac{2\Omega c}{r} = \left(DL_R + \frac{2\Omega c}{r}\right) + i\alpha_I Du$$

$$c^2 = \left(DL_R + \frac{2\Omega c}{r}\right)^2 - \alpha_I^2 Du^2 + i\alpha_I Du \left(DL_R + \frac{2\Omega c}{r}\right)$$

$$\ell = \alpha^2 + \ell^2/r^2 = \alpha_R^2 - \alpha_I^2 + \frac{\ell^2}{r^2} + 2\alpha_R \alpha_I i = \ell_R + i\ell_I$$

$$\frac{L}{A} = \frac{L_R A_R + L_I A_I}{|A|^2} , \quad \frac{(L_I A_R - A_I L_R)i}{|A|^2}$$

$$\frac{1}{AL} = \frac{1}{|AL|^2} \left\{ L_R A_R - L_I A_I \right\} - \frac{i}{|AL|^2} \left\{ L_I A_R + L_R A_I \right\}$$

$$\frac{1}{A} = \frac{1}{|A|^2} (A_R - iA_I) \qquad \frac{1}{L} = \frac{1}{|L|^2}(L_R - iL_I)$$

$$\frac{c}{A} = \frac{c_R A_R + c_I A_I}{|A|^2} + i \frac{c_I A_R - c_R A_I}{|A|^2}$$

$$\frac{c^2}{LA} = \frac{c_R^2 - c_I^2}{|LA|^2}(L_R A_R - L_I A_I) - \frac{2c_R c_I}{|LA|^2}(L_I A_R - L_R A_I)$$

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$$i\frac{(C_R^2 - C_I^2)(L_I A_R + L_R A_I)}{|LA|^2} = \frac{2C_R C_i i}{|-A|^2}(L_R A_R - L_I A_I)$$

In operator form the imaginary part of the stability integral may be written as follows:

$$\int \frac{1}{|A|^2}(L_I A_R - A_I L_R)|D\psi|^2 r \, dr + \int r|A|^2 \left\{ \frac{C_R^2 - C_I^2}{|LA|^2}(L_I A_R + L_R A_I) \right.$$

$$- \frac{2C_R C_I}{|LA|^2}(L_R A_I - L_I A_I) + (M + \frac{2}{r})\frac{(C_I A_R - C_R A_I)}{|A|^2} + L_I r \frac{L_I}{|L|^2}(\Omega_R + N')$$

$$+ \frac{L_I A_R - A_I L_R}{|A|^2}(D\ln\rho - M)(M + \frac{2}{r}) + \frac{C_I A_R - C_R A_I + 2 A_R DL_I - 2 A_I DL_R}{|A|^2}($$

$$D\ln\rho - M)\bigg\} dr + \int r \left[\frac{C_I A_R - C_R A_I}{|A|^2} + \frac{(L_I A_R - A_I L_R)(D\ln\rho - M)}{|A|^2} \right]$$

$$[\psi D^*\psi^* + \psi^* D\psi] \, dr = 0$$

In order for the above integral not to admit any α_I other than zero the sum of the above integrals cannot be zero. Assuming that the sign of the first integral is positive, the sign of the sum of the remaining integral must be positive also. This will prove that α_I can only have a zero value.

$$L_I = U\alpha_I \qquad A_I = \frac{2L_R L_I - e_I a^2}{a^2} = \alpha_I (2UL_R - 2a^2\alpha_R)$$

$$C_I = \alpha_I D u \qquad C_R = \frac{D L_R}{2} + \frac{2\Omega}{r}$$

By this deduction one can already show that α_I is a multiplier of all the above integrals. Thus, if $\alpha_I(I_1 + I_2 + I_3) = 0$ and $I_1 + I_2 + I_3 \neq 0$ then $\alpha_I = 0$. The sign of the integrals is unaffected by the factor $|A|^2$ as it is positive. Similarly:

$$L_I A_R - A_I L_R = L_I (L_R^2 - L_I^2 - e a^2)/a^2 - (2 L_R L_I - e_r a^2) L_r / a^2$$

This equation can have a zero if the multiplier is zero.

$$(L_R^2 - L_I^2 - e_r a^2) u /a^2 - (2 L_R^2 L_I - 2 d_c a^2 L_R)/a^2 = 0$$

$$- [(L_R^2 - e_r a^2)/u^2 + 2 L_R \alpha_c a^2/u^3] + 2 L_R^2 \alpha_I /u^2 + \alpha_I^2 = 0$$

The solutions for α_I for which the above argument fails are given by:

$$\alpha_I = \frac{L_R^2 M^2}{u^2(1-M^2)} \pm \sqrt{\left[\frac{L_R M^2}{u^2(1-M^2)}\right]^2 - \frac{1}{(1-M^2)}\left[d_c + \frac{L_R}{u}\right]^2 - \frac{L_R}{1-M^2}}$$

If $M^2 > 1$, the value under the square root is greater than zero and thus there are real roots for α_I. This shows that there are always unstable modes for supersonic flow based on this criterion.

$$\left\{ \frac{C_R^2 - C_I^2}{|L|^2} \frac{(L_I A_R + L_R A_I)}{(L_I A_R - A_I L_R)} - \frac{2 C_I C_R}{|L|^2} \frac{(L_R A_R - L_I A_I)}{(L_I A_R - A_I L_R)} + (M + \frac{2}{r})\left(\right.\right.$$

$$\frac{C_I A_R - C_R A_I)}{(L_I A_R - L_R A_I)} + \frac{L_I(1 + \Omega_R + N^2/L_I^2)}{(L_I A_R - A_I L_R)} + (D \ln p_o - M)(M + \frac{2}{r})$$

$$+ \frac{C_I A_R - C_R A_I + A_R D L_I - D L_R A_I}{L_I A_R - A_I L_R}(D \ln p_o - M) \geq 0$$

The condition that α_I does not satisfy the solution required to obtain a zero in the denominator, allows multiplying by $L_I A_R - A_I L_R$.

$$L_I A_R \left\{ \frac{C_R^2 - C_I^2}{|L|^2} + (D \ln p - M)(M + \frac{2}{r}) \right\} + L_R A_I \left[\frac{C_R^2 - C_I^2}{|L|^2} - (D \ln p - M)(M + \frac{2}{r}) \right]$$

$$+ C_I A_R (D \ln p + \frac{2}{r}) + C_R A_I (D \ln p + \frac{2}{r}) + |A|^2 L_I + \frac{L_I |A|^2 (\Omega_R + N^2)}{|L|^2}$$

$$+ (D \ln p - M)(A_R D L_I - A_I D L_R) \geq 0$$

The above equation may be evaluated by dividing by $L_I = \alpha_I U$ and evaluating the remaining terms:

$$(L_R^2 - L_I^2 - e_R a^2)\left[\frac{c_R^2 - c_I^2}{|L|^2} + (D\ln\rho - M)(N + 2/r) \right]/a^2 + 2c_R(L_R - \alpha_R a^2/u)\left[\right.$$

$$\frac{c_R^2 - c_I^2}{|L|^2} - (D\ln\rho - M)(M + 2/r)\right]/a^2 + \frac{Du}{u}(L_R^2 - L_I^2 - e_R a^2)(D\ln\rho + 2/r)/a^2$$

$$+ |A|^2 L_I + L_I |A|^2 (\Omega_R + N^2)/|L|^2 + 2c_R(L_R - \alpha_R a^2/u)(D\ln a + 2/r)/a^2$$

$$(D\ln\rho - M)\left(\frac{Du}{u}\Omega_R - \frac{2DL_R}{a^2}(L_R - \alpha_R a^2/u) \right)\right] \geqslant 0$$

The above equation is a rather complicated representation for the stability in space of the disturbance that was initiated at the point X = 0. In order to obtain some meaningful results the above is restricted to the axisymmetric case (i.e. ℓ = 0). Then the expressions simplify to the following:

$$L_I = U\alpha_I \; ; \; L_R = \omega + \alpha_R u \; , \quad \ell_R = \frac{\alpha_R^2 - \alpha_I^2}{r^2} \; ; \quad \ell_I = \frac{2\alpha_R \alpha_I}{r^2}$$

$$(L_R^2 - L_I^2 - e_R a^2) = \omega^2 + 2\alpha_R \omega u + \alpha_R^2(u^2 - a^2) - \alpha_I^2(u^2 - a^2)$$

$$c_R = DL_R = \alpha_R Du \qquad c_I = DL_I = \alpha_I Du$$

$$c_R^2 - c_I^2 = (\alpha_R^2 - \alpha_I^2)(Du)^2 \qquad |L|^2 = (\omega + \alpha_R u)^2 + u^2\alpha_I^2$$

$$A_I = \frac{2\alpha_R L_I}{a^2} - \frac{2\alpha_R \alpha_I a^2}{a^2} = \frac{2(\omega + u\alpha_R)\alpha_I u}{a^2} - \frac{2\alpha_R \alpha_I a^2}{a^2}$$

$$L_R A_I = \frac{2\alpha_I}{a^2}\left[\omega u + (u^2 - a^2)\alpha_R \right]\left[\omega + u\alpha_R \right]$$

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The axisymmetric perturbation criterion then becomes:

$$\frac{1}{a^2}\left[\omega^2 + 2\alpha_R u \omega + \alpha_R^2(u^2+a^2) - \alpha_I(u^2-a^2)\right]\left\{(\alpha_R^2-\alpha_I^2)\left(\frac{Du}{|L|}\right)^2 + (D\ln\rho - M)\right.$$

$$\left.(M+\tfrac{2}{r})\right\} + 2\frac{(\omega+u\alpha_R)}{a^2}\left[\omega + \frac{u^2-a^2}{u}\alpha_R\right]\left[(\alpha_R^2 - \alpha_I^2)\left(\frac{Du}{|L|}\right)^2 - (D\ln\rho - M)(M+\tfrac{2}{r})\right]$$

$$+ \frac{Du}{u}\left[\omega^2 + 2\alpha_R\omega u + (\alpha_R^2 - \alpha_I^2)(u^2-a^2)\right](D\ln\rho + 2M) + \frac{2\alpha_R Du}{a^2}\left[\omega u + (u^2-a^2)\alpha_R\right]$$

$$(D\ln\rho + \tfrac{2}{r}) + (D\ln\rho - M)\left\{\frac{Du}{u a^2}\left[\omega^2 + 2\alpha_R\omega u + (\alpha_R^2 - \alpha_I^2)(u^2-a^2)\right]\right.$$

$$\frac{2\alpha_R Du}{a^2}\left(\omega + \frac{u^2-a^2}{u}\alpha_R\right)\bigg\} + |A|^2 u + \frac{u|A|^2}{|L|^2}(\Omega_R + N^2) \geqslant 0$$

The condition of stability acquires that $\alpha_I = 0$. The above equation then becomes a function only of α_R and ω.

$$\frac{1}{a^2}\left[\omega^2 + 2\omega\alpha_R u + \alpha_R(u^2-a^2)\right]\left\{\left(\frac{Du^2}{\omega+\alpha_R u}\right)^2 + (D\ln\rho - M)(M+\tfrac{2}{r}) + \right.$$

$$\left.\frac{Du}{u}(D\ln\rho + \tfrac{2}{r}) + \frac{Du}{u}(D\ln\rho - M)\right\} + 2\left[\frac{\omega u + (u^2-a^2)\alpha_R}{u a^2}\right]\left\{\frac{\alpha_R^2 Du^2}{(\omega + \alpha_R u)^2}\right.$$

$$-\frac{(D\ln\rho - M)(M+\tfrac{2}{r})}{a^2}(u^2 + u\alpha_R) + 2\alpha_R Du(D\ln\rho + \tfrac{2}{r}) - 2\alpha_R Du(D\ln\rho$$

$$\left.- M)\right\} + |A|^2 + u\left[1 + \frac{\Omega_R + N^2}{(\omega + u\alpha_R)^2}\right] \geqslant 0$$

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The form of the criterion for this type of a perturbation history is similar to the one that is assymtotically stable for large times. The same basic parameters govern the stability; namely, the buoyancy and angular momentum given by the expressions $\Omega_R + N^2$ and the entropy given by the expression $D\ell n\rho_o - M$. The expression $M(D\ell n\rho_o - M)$ also appears. This parameter is reminiscent of the Brunt-Vaisälä buoyancy frequency and must be negative for static stability as was shown in the stratified atmospheric problem.

Also appearing explicitly for the first time is the Mach number parameter $(M^2 - 1)$. Since there is a zero crossing for $M > 1$, there may be an unstable mode associated with it for supersonic speeds. In a paper by McLaughlin et al, it was observed that there are disturbances or modes of instability present for the supersonic jet. There are also large scale structures present in the stream that were observed by Scharton and White and associated with sources of the emitted noise. This simple analysis shows that for each wave number α_R, frequency ω and steady state velocity and pressure fields, there may be instabilities in space even though instabilities in time might be ruled out.

CHAPTER 3. GENERALIZED CONFIGURATIONAL STABILITY ANALYSIS

The concept of stability concerns the behavior of an initially sta-
tionary or time independent state subjected to an arbitrary perturbation.
The system is observed after the disturbance is applied to see if it returns
to its original state. If the magnitude of the disturbing perturbation
is arbitrary, there will always be a set of perturbations greater than some
limiting value that will produce instability. A discussion of stability
which included finite perturbations to find this limiting value results in
attempting to solve a set of non-linear partial differential equations for
which even the existence of the solutions has not been proven. Therefore,
the complex equations are linearized by the assumption of infinitesimal
perturbations. These equations cannot describe the full non-linear problem,
but they can establish conditions for which the perturbations of the sta-
tionary quantities may grow in time. The non-linear terms that have been
ignored may change their behavior after the magnitude of the perturbations
has grown to finite size but for the purposes of the analysis these are
ignored. Even the linearized equations are most difficult to solve, but
they allow the use of some powerful mathematical tools to determine limit
values for the solutions. For swirling flows, this simplified method was
used by most authors with the additional assumption of a steady flow with
finite boundaries. The stationary flow was further assumed to depend only
on one space variables. Carmi[48] verified that the standard analysis of

finding a stability criteria could be extended to infinite domains provided that certain additional criteria were met. The lemmas stated further limit the problem by adding more restrictive conditions to the behavior of the steady state velocity profiles as $r \to \infty$, so that the integrals of the functions always remain finite.

The previous chapter used a modified energy approach to study the behavior of an axisymmetric flow in which the steady state variables were functions only of the distance from the center line. This is indicative of a flow in an infinite straight pipe or of a high Reynolds number jet far from the exit as was shown in the introduction. The appearance of ring vortices near the exit of a straight jet and the birth and decay of other large scale structures would indicate that transition from a top hat velocity profile at the exit to a normal distribution shape downstream might have some effect on the overall stability. Real flow fields have finite dimensions; therefore, their properties are at least weak functions of the other coordinates. For particular physical problems, the axial gradients are large, thus the analysis of chapter two may not give the true stability of such a flow system.

The stability of more general problems is derived via a further extension of the energy approach. Since for moving fluids the stability is determined by observing the behavior of the streamlines of the steady flow, defining a coordinate system based on the stationary streamlines would be advantageous. These streamlines are found by defining two stream functions Ψ_1 & Ψ_2 that satisfy the equations and boundary conditions. The gradients to surfaces of constant Ψ_i are found to determine the velocity vector:

$$\vec{V} = \frac{1}{p} \, (\text{grad } \Psi_1 \times \text{grad } \Psi_2)$$

It can be seen that the intersection of the constant surfaces of Ψ_i form a system of streamlines or flow trajectories in space. A coordinate system is established that is fixed to a particular streamline and the space variation of all flow properties are determined relative to it. This is shown in the sketch in figure 3. The perturbation quantities are defined such that each term has the form: (See the sketch for definition of ζ, t, $\vec{r}_o$, etc.)

$$P, \rho, \vec{V} = P(\vec{\xi}, t; \vec{r}_o), \; \rho(\vec{\xi}, t; \vec{r}_o), \; V(\vec{\xi}, t; \vec{r}_o)$$

All the steady state variables can be defined as referenced to this streamline, by using the initial position $\vec{r}_o$ and a vector relative to this position, $\vec{\xi}$. This vector is a function of the position on the equilibrium trajectory and of course time. Thus, the linear perturbation equations for the flow parameters at some point not on the streamline are given by:

$$\rho(\vec{r}_o + \vec{\xi}) = \rho(\vec{r}_o)(1 - \text{div }\vec{\xi})$$

$$\vec{V}(\vec{r}_o + \xi) = V(\vec{r}_o) + \vec{V}\cdot\nabla\vec{\xi} + \frac{\partial \vec{\xi}}{\partial t}$$

The remaining parameter is obtained from the energy equation and the assumption of an initially isotropic flow.

$$\frac{DP}{Dt} = a^2 \frac{D\rho}{Dt} \; \Rightarrow \; \frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right) = 0$$

This equation gives the perturbation for P to be:

$$P(\vec{r}_o + \vec{\xi}) = P(\vec{r}_o)(1 - \gamma\nabla\cdot\vec{\xi})$$

The gradient operator is given at $\vec{r}_o + \vec{\xi}$ by the expression $\Delta = \Delta^o - \Delta\vec{\xi}\cdot\Delta^o$.

The steady state quantities are evaluated at $\vec{r}_o$ along the equilibrium trajectory to obtain an equation that depends only on the local displacements. In the Lagrangian representation, referring the steady state quantities to $\vec{r}_o$ makes the steady state properties depend only on $\vec{r}_o$. These definitions are substituted into the time dependent equations of motion as outlined by equations (3-1) through (3-3). The equations are written with the material derivative because it is defined as the derivative along a path line. With the equations of motion in divergence form, it is possible to see that they concern the local and convective changes along a stream line.

Continuity:
$$\frac{D\rho}{Dt} + \rho \, div \, \vec{v} = 0 \qquad (3-1)$$

Momentum:
$$\frac{\partial \rho \vec{v}}{\partial t} + div \left[\rho \vec{v} \vec{v} + P \underset{\sim}{I} \right] = 0 \qquad (3-2)$$

Energy:
$$\frac{DP}{Dt} = a^2 \frac{D\rho}{Dt} \qquad (3-3)$$

By focusing our attention to what happens to an individual fluid particle in the course of time, it is possible to determine if a particular flow configuration is stable or not. The motion of each fluid particle is a function of the particle parameters and time. Once the properties of the flow field have been given as a function of position, the position vector $\vec{r}_o$ describes all the properties of the particle. Therefore, in this representation the location of the fluid element is the governing variable in determining the behavior of the flow:

$$\vec{r} = \vec{r}_o + \vec{\xi} \, (\vec{r}_o, t)$$

The vector $\vec{r}_o$ refers to the position along the equilibrium trajectory at time, $t = t_o$. At this time $\vec{\xi} = 0$. At some later time the particle is given by the vector $\vec{r}^o$ and $\xi(\vec{r}^o, t)$ where $\vec{r}^o$ is the position it would have on the equilibrium trajectory and $\vec{\xi}(\vec{r}^o, t)$ is the distance to the displaced trajectory. By using the representation that $\vec{r}_o$ and $\vec{r}^o$ depend on a particular particle of a fluid, the equations assume the Lagrangian representation with the steady state variables independent of time. Therefore, the solution of the problem is the position of the fluid element as a function of time. The equation may also be put into an Euler representation by evaluating the flow properties at each of the positions $\vec{r}_o$ and $\vec{r}^o$ for all trajectories and as functions of the space variables.

These definitions apply only for small perturbations, and therefore, are useable to find the hydrodynamic stability of stationary flows subject to infinitesimal disturbances. The growth and decay of the perturbations for large times determines the character of the stability of the system. Stability thus can be determined for a problem by studying the assymtotic behavior of the solution to the stability equation. This allows one to establish stability limits or give a more general stability criteria in an analytical manner, even though the general or a particular solution of the governing equation is not known.

For most problems of interest, the method is possible because the linearized equations are used. This is why much of the work of hydrodynamicists has been through the analysis of these linearized equations. With respect to their validity it can be stated that if the perturbations are small the non-linear terms are of second orders and may be neglected. This is true

even though the non-linear terms involve derivatives and these in general
are not small, but there is sufficient experimental evidence that their
elimination is supported by the available data.

The application of the Euler equations (3-1) to (3-3) involves a fur-
ther assumption that the effects of the viscosity and heat conduction are
not important to the initial growth and decay of the perturbations. This
is assumed to be true because the flows that are considered to have small vis-
cosity behave nearly inviscidly except at a critical point $\vec{r} = \vec{r}_c$ where
the disturbance propagation speed equals the local magnitude of the velocity.
As far as the amplified disturbances are concerned, the effect of the fric-
tional forces is negligible in the limit $r \to r_c$. It is also known that
neutral disturbances satisfy the inviscid equations, but that the singularity
at the critical point must be examined more carefully so that the solution
remains on the same branch on either side of the singularity. Damped dis-
turbances require the introduction of the dissipative terms, but since the
limits of stability are obtained in this text by the exclusion or non-exis-
tence of amplified disturbances, the dissipative terms may be dropped. The
neutral stability limit so obtained is then a sufficiency condition that
amplified disturbances will not exist.

Substitution of the assumed perturbation quantities into the full
momentum equation (3-2) gives the equation for stability. First, evaluating
the substantial derivatives one has:

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial y_\alpha} \frac{\partial y_\alpha}{\partial x_i} \Rightarrow \frac{\partial F}{\partial y_\alpha} = \frac{\partial x_i}{\partial y_\alpha} \frac{\partial F}{\partial x_i} \qquad \text{but } y_\alpha = x_i + \zeta_\alpha (x_j, t)$$

$$\frac{\partial y_\alpha}{\partial x_j} = \frac{\partial x_\alpha}{\partial x_j} \quad , \quad \frac{\partial \xi_\alpha}{\partial x_j} = \delta_{\alpha j} + \frac{\partial \xi_\alpha}{\partial x_j} \quad \text{for small } \alpha \quad \frac{\partial x_i}{\partial y_\alpha} = \delta_{i\alpha} - \frac{\partial \xi_i}{\partial y_\alpha}$$

$$= \delta_{i\alpha} - \frac{\partial \xi_i}{\partial x_j} \frac{\partial x_j}{\partial y_\alpha}$$

$$\frac{\partial x_i}{\partial y_\alpha} = \delta_{i\alpha} - \frac{\partial \xi_i}{\partial x_j}\left(\delta_{j\alpha} + O(\xi)\right) + O(\xi^2) \Rightarrow \frac{\partial F}{\partial y_\alpha} = \frac{\partial F}{\partial x_i}\left(\delta_{i\alpha} - \frac{\partial \xi_i}{\partial x_\alpha}\right)$$

$$= \text{grad } F - \text{grad }\vec\xi : \text{grad } F$$

The time derivative is given by:

$$\frac{\partial F}{\partial t}\bigg|_{y} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y_\alpha}\frac{\partial y_\alpha}{\partial t} = \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial x_\alpha} - \frac{\partial \xi_i}{\partial x_\alpha}\frac{\partial F}{\partial x_i}\right)\frac{\partial \xi}{\partial t}$$

$$= \frac{\partial F}{\partial t}\bigg|_{x_j} - \frac{\partial F}{\partial x_\alpha}\frac{\partial \xi}{\partial t} = \frac{\partial F}{\partial t} - \text{grad } F \cdot \frac{\partial \vec\xi}{\partial t}$$

$$\frac{D}{Dt}\bigg|_{x_j} = \frac{\partial}{\partial t} + \frac{\partial \vec\xi}{\partial t}\cdot\nabla + \left(\vec v + \vec v\cdot\nabla\vec\xi + \frac{\partial\vec\xi}{\partial t}\right)\left(\nabla\cdot - \nabla\vec\xi\cdot\nabla^i\right)$$

$$= \frac{\partial}{\partial t} + \vec v \nabla + O(\xi^2)$$

$$\frac{D\vec v}{Dt} = \frac{\partial^2\vec\xi}{\partial t^2} + 2\vec v\,\nabla\frac{\partial\vec\xi}{\partial t} + \vec v\cdot\nabla\vec v + \vec v\,\nabla(\vec v\cdot\nabla\vec\xi) + O(\xi^2)$$

$$\nabla P = \left(\nabla^i - \nabla\vec\xi\cdot\nabla\right)\left[P(\vec r_o)\left(1 - \gamma\,\text{div}\,\vec\xi\right)\right] =$$

$$\nabla P(\vec r_o) - \gamma\,\nabla(P(\vec r_o)\,\text{div}\,\vec\xi) - \nabla\vec\xi\cdot\nabla P$$

The momentum equation at $\vec r + \vec\xi$ is then given by:

$$\rho\frac{\partial^2\vec\xi}{\partial t^2} + 2\rho\vec v\cdot\nabla\frac{\partial\vec\xi}{\partial t} + \rho\vec v\cdot\nabla(\vec v\cdot\nabla\vec\xi) - \rho\vec v\cdot\nabla\vec v\,\text{div}\,\xi \qquad (3\text{-}4)$$

$$- \gamma\nabla(P(\vec r_o)\,\text{div}\,\vec\xi) - \nabla\vec\xi\cdot\nabla P$$

Making use of the momentum equation and the equation:

$$\nabla\vec\xi\cdot\nabla P = \nabla(\vec\xi\cdot\nabla P) - \vec\xi\cdot\nabla(\nabla P) + \nabla P \times (\nabla\times\vec\xi)$$

$$= \nabla(\vec\xi\cdot\nabla P) + \vec\xi\cdot\nabla(\rho\vec v\cdot\nabla\vec v) + \nabla P\times(\nabla\times\vec\xi)$$

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\rho \vec{V} \cdot \nabla \frac{\partial \vec{\xi}}{\partial t} + \nabla \cdot \left[\rho \vec{V} \vec{V} \cdot \nabla \vec{\xi} - \rho \vec{\xi} \, \vec{V} \cdot \nabla \vec{V} \right] = \nabla (\partial P \, div\vec{\xi} +$$

$$\vec{\xi} \cdot \nabla P) + \nabla P \times (\nabla \times \vec{\xi})$$

Thus the equation has the form:

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\rho \vec{V} \cdot \nabla \frac{\partial \vec{\xi}}{\partial t} - F(\vec{\xi}) = 0$$

where $F(\xi)$ is the sum of the pressure and inertial terms.

The above equation shows some important differences from the equations of Solov'ev[42] and Frieman and Rotenberg[43], that is, the appearance of the term $\Delta P \times (\Delta \times \vec{\xi})$. If it can be shown that the pressure gradient and the curl of the displacement vector are parallel, their equation is recovered.

Equation (3-4) is the linearized perturbation equation that must be evaluated to determine the stability of a stationary flow with the steady state variable P, ρ and $\vec{V}$ specified. The equation is a complex vector equation that can be reduced to a scalar real equation by multiplying it with ξ^*, the complex conjugate of ξ, and integrating over an appropriate volume.

We will begin by examining the convective and pressure terms. The expression $\vec{\xi}^* \cdot \text{grad} \left[\gamma P div\vec{\xi} \right]$ may be replaced by the equivalent expression $\text{div} \left[\gamma P \vec{\xi}^* \cdot div\vec{\xi} \right] - \gamma P |div\vec{\xi}|^2$. The total divergence integrates to zero over a domain with ξ_n equal to zero on the boundary, and the remaining term is equal to the potential energy of a compressible fluid element that oscillates as it travels along the trajectory.

It is possible to evaluate the pressure and convective terms by expanding the divergence term in the equation of stability and by evaluating the

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gradient of the pressure in terms of the convective momentum. Thus:

$$\text{div}\left[\rho\vec{\xi}\,(\vec{v}\cdot\nabla\vec{v}) - \rho\vec{v}\,(\vec{v}\cdot\nabla\vec{\xi})\right] = \text{div}(\rho\vec{\xi}\,\vec{a} - \rho\vec{v}\,\vec{b}) = \rho\vec{\xi}\cdot\nabla\vec{a} + \text{div}\,\rho\vec{\xi}\,\vec{a} - \rho\vec{v}\cdot\nabla\vec{b}$$

The above terms are multiplied by $\vec{\xi}^{*}$ and interpreted over the boundary with $\xi^{n} = 0$. The pressure term is:

$$\vec{\xi}^{*}\cdot\text{grad}(\vec{\xi}\cdot\nabla P) = \text{div}\left[\vec{\xi}\,(\vec{\xi}\cdot\nabla P)\right] - \vec{\xi}\cdot\nabla P\,\text{div}\,\vec{\xi}$$

The steady state momentum equation is given by $\rho\vec{a} = -\Delta P$. This leads to the equation:

$$2\,\text{Re}\left[\rho\vec{a}\vec{\xi}^{*}\text{div}\vec{\xi}\right] + (\vec{a}\cdot\vec{\xi}^{*})(\vec{\xi}\cdot\text{grad}\,\rho)$$

The expression $\vec{\xi}^{*}\cdot(\rho\vec{V}\cdot\Delta\vec{b})$ may be integrated by parts and using the fact that div $\rho\vec{V} = 0$ yields:

$$\int \vec{\xi}_{i}^{*}\,\rho v_{j}\,\frac{\partial b_{i}}{\partial x_{j}}\,dx = \int \vec{\xi}_{i}^{*}\,\frac{\partial}{\partial x_{j}}(\rho v_{j}b_{i})\,dx = \int_{s} \rho v_{j}\,b_{i}\,\vec{\xi}_{i}^{*}n_{i}\,ds -$$

$$\int \rho v_{j}\,b_{i}\,\frac{\partial\xi_{i}}{\partial x_{j}}\,dx = -\int \rho|\vec{b}|^{2}dx$$

This leaves the terms $\rho\vec{\xi}^{*}\cdot(\vec{\xi}\cdot\Delta\vec{a}) + (\vec{\xi}^{*}\cdot\vec{a})(\vec{\xi}\cdot\text{grad}\,\rho)$ unaccounted for. The last term is part of a vector cross product. Expand $(\vec{\xi}\cdot\Delta\rho\vec{a})$ in terms of factors.

$$\vec{\xi}\cdot\nabla\rho\vec{a} = \frac{1}{2}\left\{\text{grad}\,(\rho\vec{a}\cdot\vec{\xi}) - \vec{\xi}\times\text{curl}\,\vec{a} - \text{curl}\,(\vec{\xi}\times\rho\vec{a}) + \vec{\xi}\,\text{div}\,\rho\vec{a}\right.$$

$$\left. - \rho\vec{a}\,\text{div}\,\vec{\xi}\right\}$$

$$= \frac{\rho}{2}\left\{\text{grad}\,(\vec{a}\cdot\vec{\xi}) - \vec{\xi}\times\text{curl}\,\vec{a} - \vec{a}\times\text{curl}\,\vec{\xi} - \text{curl}\,(\vec{\xi}\times\vec{a})\right.$$

$$\left. \vec{\xi}\,\text{div}\,\vec{a} - \vec{a}\,\text{div}\,\vec{\xi}\right\} + \frac{1}{2}\left\{(\vec{a}\cdot\vec{\xi})\,\text{grad}\,\rho - \vec{\xi}\times(\text{grad}\,\rho\times\vec{a})\right.$$

$$\left. - \text{grad}\,\rho\times(\vec{\xi}\times\vec{a}) + \vec{\xi}\,(a\cdot\text{grad}\,\rho)\right\}$$

$$= \rho\vec{\xi}\cdot\nabla\vec{a} + (\vec{\xi}\cdot\text{grad}\,\rho)\vec{a}$$

Since time does not appear explicitly in the above equation and since

the equations are linear in the dependent variable $\vec{\xi}$, it is possible to

eliminate the dependence on time by use of either a transformation on time

or by assuming that the time dependence is separable and can be written in

the form $\vec{\xi} = e^{-i\omega t} \vec{\xi}(\vec{r})$. This changes the form of equation (3-4):

$$- \rho \omega^2 \vec{\xi} + 2i\omega \, \rho \vec{v} \cdot \nabla \vec{\xi} - F(\vec{\xi}) = 0 \qquad (3\text{-}5)$$

This equation is multiplied by $\vec{\xi}^*$ and integrated over all $\vec{r}_0$ to obtain

the equation as indicated below:

$$- \omega^2 \int \rho |\vec{\xi}|^2 d\tau + 2i \int i\omega(\, \rho \vec{v} \cdot \nabla \vec{\xi}) \cdot \vec{\xi}^* d\tau - \int \vec{\xi}^* \cdot F(\vec{\xi}) d\tau = 0 \qquad (3\text{-}7)$$

Integration by parts shows that the second is real. Thus:

$$\int 2i \, (\rho \vec{v} \cdot \nabla \vec{\xi}) \cdot \vec{\xi}^* d\tau = 2 Re \int i \, (\rho \vec{v} \cdot \nabla \vec{\xi}) \cdot \vec{\xi}^* d\tau$$

Stability depends on the growth of the arbitrary disturbance $\vec{\xi}$ with

time. This condition is satisfied only if ω is real. Equation (3-6) is

quadratic and can be solved for ω. Only the solution for large time is

required, thus all the needed information is obtained from the sign of the

discriminant.

$$\left[\int 2i (\rho \vec{v} \cdot \nabla \vec{\xi}) \cdot \vec{\xi}^* d\tau \right]^2 + 4 \int \rho |\vec{\xi}|^2 d\tau \int F(\vec{\xi}) \cdot \vec{\xi}^* d\tau \geq 0 \qquad (3\text{-}7)$$

In order for a steady fluid dynamic system to have internal stability

it must satisfy the steady state equation and the stability equation (3-5).

The stability equation deals with the behavior of the steady state fluid

elements subjected to infinitesimal perturbations. Apparent changes in the

motion will be observed if the particles do not return to their initial paths but to new ones that can satisfy both the steady state and perturbation equations. Changes of this kind can be observed by the introduction of disturbances into flows that result in the appearance of vortices in an irrotational flow due to the pressence of a lifting wing, the formation of whirlpools and other similar phenomena. In most of these there is an initial displacement of the streamlines such that $\vec{V} \cdot \vec{\xi} = 0$. Perturbations that satisfy this criterion are the ones that introduce rotation into the system. For a perturbation that is parallel to the velocity vector $\vec{V} \times \vec{\xi} = 0$. This would leave only the gradient terms in the perturbation equation, and this problem would reduce to potential oscillations about a given streamline. The remaining terms act like a potential function. Thus, stability is achieved if the potential energy attains an absolute minimum or a local minimum for infinitesimal oscillations. These are not as interesting as the others that occur in nature, so in this section only the perturbations normal to a stream surface will be considered.

For disturbances whose normal vanishes on the boundary, the integral of $\vec{\xi}^* \cdot F(\vec{\xi})$ is real because the operator is self-adjoint. The first term in the discriminant is real and positive for all velocities $\vec{V} \neq 0$. If $\vec{V} = 0$, the stability condition reduces to determining that the potential energy given by $\frac{1}{2} \int \vec{\xi}^* \cdot F(\vec{\xi}) dt > 0$ for all arbitrary displacements $\vec{\xi}$ that vanish on the chosen boundary for the steady state flow.

The integrand $\vec{\xi}^* \cdot F(\vec{\xi})$ is given by:

$$\vec{\xi}^* \cdot F(\vec{\xi}) = \gamma P |div\, \vec{\xi}|^2 - 2\rho \vec{v} \cdot \nabla \vec{v}\, R_e\, (\vec{\xi}^* div\, \vec{\xi}) + \qquad (3\text{-}8)$$

$$(\vec{v} \cdot \nabla \vec{v} \cdot \vec{\xi}^*)(\vec{\xi} \cdot grad\, p) - \rho |\vec{Q}|^2 - \rho |\vec{\xi}|^2 \frac{1}{h_1 h_2 h_3} \left[u \left(h_3 \frac{\partial h_3}{\partial \xi_1} - h_2 \frac{\partial h_2}{\partial \xi_1} \right) + \frac{\partial u}{\partial \xi_1} \right]$$

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$$\frac{\partial}{\partial \xi_1}\left[h_2 h_3 u\right] + \frac{\partial}{(h_2 h_3)^2}\left|\frac{\partial}{\partial \xi_3}(h_2 u_3)\right|^2 + 2\rho u|\vec{3}|^2 \frac{1}{h_1 h_2 h_3^2} \frac{\partial h_1}{\partial \xi_3}$$

$$\frac{\partial u h_2}{\partial \xi_3} + 2\frac{\rho u^2}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} Re\left(\vec{3}^* \frac{\partial \vec{3}}{\partial \xi_3}\right)$$

The condition for stability given by equation (3-7) may be simplified slightly by assuming that the flow is between two stream surfaces separated by a small distance δ. It is known that the first term in the stability condition is positive by definition. Therefore, a restrictive condition for stability is to assume that the remaining integral over all applicable space must be positive for the square root to have only positive

It is the purpose of this section to show that there are more general conditions than the ones derived that allow the steady flow field to depend only on one of the coordinates.

Consider a flow between two closely spaced stream surfaces for which the distance between them depends on the axial distance. If the velocity vector is not alligned with the axial direction, then both metrics h_1 and h_3 must be functions of the axial distance. The metric h_3 is a constant to the order of the approximation. Equation (3-8) becomes:

$$\vec{3}^* \cdot F(\vec{3}) = 3P|div\vec{3}|^2 + \frac{\rho u^2}{3Ph_1}\frac{\partial h_1}{\partial \xi_3}\vec{3}|^2 - \frac{u^2}{h_1}\frac{\partial h_1}{\partial \xi_3}\frac{\partial \rho}{\partial \xi_3}|\vec{3}|^2 - \rho|\vec{a}|^2 \tag{3-9}$$

$$\rho|\vec{3}|^2\frac{1}{h_1 h_3^2}\left|\frac{\partial h_2 u}{\partial \xi_1}\right|^2 + \frac{\partial}{h_2}\left|\frac{\partial}{\partial \xi_3}(u h_2 \vec{3})\right|^2 + 2\frac{\partial u}{h_1 h_2}|\vec{3}|^2 \frac{\partial h_1}{\partial \xi_3}\frac{\partial u h_2}{\partial \xi_3}$$

$$+ 2\frac{\rho u^2}{h_1}\frac{\partial h_1}{\partial \xi_3} Re\left(\vec{3}^* \frac{\partial \vec{3}}{\partial \xi_1}\right) - \frac{1}{3P}\left(\frac{\rho u^2}{h_1}\frac{\partial h_1}{\partial \xi_3}\right)^2|\vec{3}|^2$$

It was shown earlier in Chapter three that the basic perturbation displacement equation for an inviscid, non-heatconducting, compressible flow may be written

$$\rho \left[\frac{\partial^2 \vec{\xi}}{\partial t^2} \right] + 2\rho \vec{v} \cdot \nabla \frac{\partial \vec{\xi}}{\partial t} + \rho \vec{v} \cdot \nabla (\vec{v} \cdot \nabla \vec{\xi}) - \rho \vec{v} \cdot \nabla \vec{v} \, div \, \vec{\xi} = \nabla (c^2 \rho (div \, \vec{\xi})) - \nabla \vec{\xi} \cdot \nabla \rho$$

In order to obtain the above it was assumed that the relationship between pressure and the density is given by

$$\frac{dP}{dt} = \frac{1}{a^2} \frac{d\rho}{dt} \Rightarrow \frac{1}{P} \frac{dP}{dt} = \frac{\gamma}{\rho} \frac{d\rho}{dt} \quad or \quad \frac{d}{dt}\left(\frac{P}{\rho^\gamma} \right) = 0$$

The above equation is valid for these flows for which it is possible to divide the speed of sound and integrate the above energy equation directly. An integral for the momentum equation exists.

The second law of thermodynamics states that

$$T \, ds = de + P \, dv = dh - v \, dP$$

If the fluid is assumed to be thermally and calorically perfect, the above integrates to:

$$P = \rho^\gamma h \, exp \, (-s/c_v)$$

The equation of motion given above states that the quantity P/ρ^γ must be constant along a streamline for the result to apply. The remaining condition, $ds/dt = 0$, states that the exponential is a constant in the direction of the streamline. This is equivalent to stating that unless there is a heat loss or gain associated with the motion of the fluid element, its entropy remains constant. Therefore, under the assumptions used to construct the stability equation, entropy gradient normal to the direction of motion are acceptable.

The coordinate system as defined relates the properties of the trajectory. The velocity vector is in the direction of the tangent vector, the other vector in the stream surface is in the direction of the principle normal line and the $\vec{\xi}$ vector is in the direction of the binormal. This means that the derivatives of the unit vector satisfy the Frenet equations. Thus, only particular combinations of derivatives of the unit vectors are required to describe the behavior of the curve and thus the derivatives of the metrics must be related. The describing quantities are the curvatures and the torsion of the velocity curve.

The first term in the expression of stability (equation 3-7) is positive; therefore, a more restrictive stability condition is to assume that $\int \vec{\xi}^{*} \cdot F(\vec{\xi}) d\tau > 0$. This also allows examining disturbances that are real only. The above is the general stability equation for arbitrary small disturbances of a flow with $\vec{\xi} = 0$ on the boundary.

The above expression is real as was shown by the Hermitian property of the inviscid operator. The problem must be examined on a term by term basis to find the criteria for a particular flow. Examination of the further details of this problem will be left to the discussion in chapter 4.

Up to this juncture we have dealt only with the stability of flows that satisfied the inviscid Euler equations, and have restricted the analysis to those systems for which amplified or neutrally stable perturbations are possible. The effects of viscous dissipation do not always lead to greater stability compared with the inviscid flow as was shown by Tollmein[3] for boundary layer type flows and by Yih[33] for the narrow gap rotational

Couette flow. In order to analyze the effects that the frictional forces will have on the disturbances, it is necessary to include them when deriving the equations for stability. For incompressible media the work of Reynolds and later on Orr gave the fundamental results that the viscous forces can be important when compared to the inertial force. The stability limits obtained experimentally were later computed for simple incompressible flows in a more rigorous mathematical manner by Serrin[50], Prodi[15], Velte[11], and Sattinger[3] among others. These were based on the growth of an infinitesimal velocity disturbance. Their stability equation determined the energy difference function, K, via the integral equation:

$$\frac{dK}{dt} = - \int (\vec{u} \cdot \underline{D} \cdot \vec{u} + \nu \nabla \vec{u} \cdot \nabla \vec{u}) \, dc \qquad (3\text{-}10)$$

It can readily be seen that extending the energy difference integral to include full compressibility would be a formidable task. Another approach is the configurational stability method that reduces the number of unknown dependent variables to only the displacement vector, $\vec{\xi}$, relative to the equilibrium trajectory. It is more suited to determine the stability of a system to small disturbances. The difference of energy is a positive quantity; thus the condition $\frac{dK}{dt} \leq 0$ assures that $K \to 0$ as $t \to \infty$. The fully compressible viscous problem of configurational stability is treated similarly except that the conservation form of the equations are used to eliminate the perturbation equations as a function of the displacement of a particle and the integration takes place over all the individual particles along an equilibrium trajectory. These fluid elements are defined by their position vectors $\vec{r}$ and $\vec{\xi}$, where $\vec{r}_o$ depends on the particle. The perturbation equations are then obtained and an energy integral approach is developed.

Because the addition of viscosity the entropy production now changes
the energy equation to:

$$\rho T \frac{ds}{dt} = \underset{\sim}{\tau} : \text{grad } u - \text{div } \vec{q} \qquad (3\text{-}11)$$

where

$$\underset{\sim}{\tau} = \mu \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \text{is a symmetric tensor.}$$

If the heat transfer rate is small, then the entropy production mechan-
isms are the shear stresses. The energy equation has an integral for Prandtl
number equal to one that eliminates the need for a complicated representa-
tion of the relationship among all the thermodynamic variables. This inte-
gral requires that the total temperature be constant. This assumption is
valid approximately for more complicated flow fields also and is given as
follows:

$$c_p T + \tfrac{1}{2} u^2 = \text{const} = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} + \tfrac{1}{2} \vec{u} \cdot \vec{u} \qquad (3\text{-}12)$$

One may also assume a thermal and calorically perfect gas and use the
thermodynamic pressure-density relationship to derive a relationship that
can account for entropy changes.

$$\frac{P_2}{\rho_2} = \frac{P_1}{\rho_1} \exp \left(-\frac{\Delta S}{R} \right) \approx \left(\frac{\rho_2}{\rho_1} \right)^C \quad \text{or} \quad \frac{dP}{dt} - a^2 \frac{d\rho}{dt} = \underset{\sim}{\tau} \cdot \text{grad } \vec{u}$$

where the exponent C is now governed by the magnitude of the entropy
production. Thus:

$$P + dP = (\rho + d\rho)^C = \rho^C \left(1 + C \frac{d\rho}{\rho} \right)$$

$$dP = -C \, \text{div } \vec{\xi} \, P \qquad (3\text{-}13)$$

The perturbation quantity defined via equation (3-12) is given by:

$$\frac{\gamma (P + P')}{(\gamma - 1)(\rho + \rho')} + \tfrac{1}{2} \left(\vec{V} + \vec{V} \cdot \nabla \vec{\xi} + \frac{\partial \vec{\xi}}{\partial t} \right) \cdot \left(\vec{V} + \vec{V} \cdot \nabla \vec{\xi} + \frac{\partial \vec{\xi}}{\partial t} \right) = 0$$

$$\frac{\gamma}{\gamma-1} \frac{P}{\rho}\left[1 + \frac{P'}{P} - \frac{\rho'}{\rho}\right] + \frac{1}{2}\left[\vec{V}\cdot\vec{V} + 2\vec{V}\cdot(\vec{V}\cdot\nabla\vec{\xi}) + 2\vec{V}\cdot\frac{\partial\vec{\xi}}{\partial t}\right] = 0$$

then:
$$\frac{\gamma}{\gamma-1}\left(\frac{P'}{P} - \frac{\rho'}{\rho}\right)\frac{P}{\rho} + \vec{V}\cdot(\vec{V}\cdot\nabla\vec{\xi}) + \vec{V}\cdot\frac{\partial\vec{\xi}}{\partial t} = 0 \qquad (3\text{-}14)$$

The perturbation of the density is determined from the principle of conservation of mass and the pressure perturbation is determined from the conservation of energy.

$$\frac{1}{2}\vec{V}\cdot\vec{V} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \frac{1}{2}\left(\vec{V} + \vec{V}\cdot\nabla\vec{\xi} + \frac{\partial\vec{\xi}}{\partial t}\right)\cdot\left(\vec{V} + \vec{V}\cdot\nabla\vec{\xi} + \frac{\partial\vec{\xi}}{\partial t}\right) +$$
$$\frac{\gamma}{\gamma-1}\frac{P+P'}{\rho+\rho'}$$

$$\frac{P'}{P} = -\frac{\gamma-1}{a^2}\left(\vec{V}\cdot\nabla\vec{\xi}\cdot\vec{V} + \vec{V}\cdot\frac{\partial\vec{\xi}}{\partial t}\right) - \operatorname{div}\vec{\xi}$$

The viscosity perturbation depends on the temperature of the medium.

$$\frac{\mu+\mu'}{\mu} = \frac{T+T'}{T} \longrightarrow \frac{\mu'}{\mu} = \frac{T'}{T}$$

The temperature perturbation is obtained from the equation of state:

$$\frac{P+P'}{P} = \frac{(\rho+\rho')(T+T')}{\rho T}$$

hence:

$$\frac{P'}{P} = \frac{\rho'}{\rho} + \frac{T'}{T}$$

From the energy equation the above formula becomes:

$$\frac{T'}{T} = -\frac{\gamma-1}{a^2}\left(\vec{V}\cdot\nabla\vec{\xi}\cdot\vec{V}\right) - \frac{\gamma-1}{a^2}\vec{V}\cdot\frac{\partial\vec{\xi}}{\partial t}$$

The equation of motion contains the divergence of the sheer stress. Up to this point, the divergence operator has not been defined. The continuity equation also contains the divergence operator but all the remaining terms have been previously defined. The continuity equation is written as the sum of the substantial derivative of ρ' plus $\rho\operatorname{div}\vec{V}$. The substantial derivative has been evaluated to give:

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\Big|_{\vec{r_i}+\vec{\xi}} = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\Big|_{\vec{r_i}}$$

This gives for the continuity equation:

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\rho + \rho\, div\,\vec{v} = 0$$

which is evaluated at $\vec{r}_0 + \vec{\xi}$:

$$-\rho\frac{\partial}{\partial t}div\,\vec{\xi} + \vec{v}\cdot\nabla\rho - \vec{v}\cdot\nabla(\rho\,div\,\vec{\xi}) + \rho\,div'\vec{v} - \rho\,div\,\vec{\xi}\,div\,\vec{v} + \rho\,div\left(\vec{v}\cdot\nabla\vec{\xi} + \frac{\partial\vec{\xi}}{\partial t}\right) = 0$$

The above equation may also be written:

$$div\left(\rho\vec{v}\,div\,\vec{\xi}\right) + \rho\,div'\vec{v} + \rho\,div(\vec{v}\cdot\nabla\vec{\xi}) = 0$$

$$div\left(\rho\vec{v}\,div\,\vec{\xi}\right) = div\,\rho\vec{v}\,div\,\vec{\xi} + \rho\vec{v}\cdot grad(div\,\vec{\xi})$$

$$v_j\frac{\partial}{\partial x_j}\left(\frac{\partial\xi_i}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(v_j\frac{\partial\xi_i}{\partial x_j}\right) = v_j\frac{\partial}{\partial x_j}\left(\frac{\partial\xi_i}{\partial x_i}\right) - \frac{\partial v_j}{\partial x_i}\frac{\partial\xi_i}{\partial x_j} - v_j\frac{\partial\xi_i}{\partial x_i\partial x_j}$$

This leaves for the divergence operator:

$$div' = -\nabla\vec{\xi}:\nabla$$

We consider a continuum with identifiable material elements that fill a volume τ_0 at time t_0. At some later time t they will occupy the volume τ. The volume element will be displaced from the initial position. In the course of the displacement the volume element will be displaced relative to some reference element within the volume. If we now assume that the displacement takes place along a path displaced from the original by the vector $\vec{\xi}(\vec{r}_0, t)$. The new deformation tensor will depend on the relative distortion of the volume element. A material point defined at time, t_0,

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relative to a globally defined cartesian coordinate system is given by:

$$\vec{r_o} = c_i \, y^i_o = c_i \, y^i(x^i, t_o)$$

At some later time this point will be given by the vector:

$$\vec{r} = c_i \, y^i(x^i, t)$$

The base vectors in the coordinate system moving with the volume is given by:

$$\vec{b_j} = \frac{\partial \vec{r}}{\partial x_j} = c_i \, \frac{\partial y^i(\vec{x}, t)}{\partial x_j}$$

and in the initial frame they are:

$$\vec{a_j} = \frac{\partial \vec{r_o}}{\partial x_j} = c_i \, \frac{\partial y^i(x, t_o)}{\partial x_j}$$

In the above, $\vec{C_i}$ are the base vectors in the cartesian reference system fixed with the space. In the displaced coordinate system the vector $\vec{r} + \vec{\xi}$ is given by:

$$\vec{d_j} = c_i \, y^i(\vec{r} + \vec{\xi}, t) = c_i \, \frac{\partial y^i(\vec{r} + \vec{\xi}, t)}{\partial x_j}$$

In analyzing the displacement three coordinate systems are needed to define the position of a material point P. The coordinate system fixed to the point P is a curvilinear system whose deformation defines the rate of strain tensor.

Now define, a, the local distance relative to P by P_o'. Then the vector $\vec{P_o P_\gamma} = d\vec{r_o}$ is represented by the form: $\quad d\vec{r_o} = \vec{a_i} \, dx^i$

and the length of the elements is:

$$(ds)^2 = d\vec{r_o} \cdot d\vec{r_o} = \vec{a_i} \cdot \vec{a_j} \, dx^i dx^j$$

where $a_i a_j$ are the metric coefficients of the curvilinear coordinate system at t_o. Similarly it is possible to form the vector relative to $\vec{r}$ by:

$$(d\dot{s})^2 = \vec{b_i} \cdot \vec{b_j} \, dx^i dx^j = g_{ij} \, dx^i dx^j$$

The displaced point in the moving coordinate system is defined by:

$$(ds)^2 = \vec{d_i} \cdot \vec{d_j} \, dx^i dx^j$$

Let $\vec{\xi} = \vec{r}_1 - \vec{r}$, denote the components of $\vec{\epsilon}$ relative to the basis $\vec{b}$ by u^i and components relative to the basis $\vec{d}_i$ by w^i. Then:

$$\vec{\xi} = u^i \vec{b}_i \qquad \vec{\xi} = w^i \vec{d}_i$$

Then the derivative of $\vec{\epsilon}$ is given by:

$$\frac{\partial \xi^i}{\partial x^j} = \frac{\partial r_i'}{\partial x_j} - \frac{\partial r^v}{\partial x_j} = d_i - b_i$$

Hence as was shown much earlier:

$$d_i = b_i + \frac{\partial \vec{\xi}}{\partial x^i}$$

But
$$g_{ij} = d_i \cdot d_j = \left(b_i + \frac{\partial \xi^i}{\partial x_j} \right) \left(b_j + \frac{\partial \xi}{\partial x_i} \right)$$

The perturbation strain given by: $g'_{ij} - g_{ij} = 2E'_{ij}$ is given by:

$$2 E_{ij} = \frac{\partial \vec{\xi}}{\partial x_i} \cdot \frac{\partial \vec{\xi}}{\partial x_j} + b_i \cdot \frac{\partial \vec{\xi}}{\partial x_j} + b_j \cdot \frac{\partial \vec{\xi}}{\partial x_i}$$

We have already demonstrated that the left hand side of the equation is self-adjoint. The shear terms give the result when multiplied by $\vec{\xi}_j^*$ the solution of the adjoint equations.

$$\int \vec{\xi}^* \cdot \left\{ \nabla \cdot \left(\mu \nabla \frac{\partial \vec{\xi}}{\partial t} + \mu \vec{v} \cdot \nabla \nabla \vec{\xi} \right)_{sym} - \nabla \vec{\xi} : \nabla \underset{\sim}{\zeta} + \nabla \vec{\xi} \cdot \nabla \cdot \underset{\sim}{\zeta} + \frac{\mu'}{\mu} \underset{\sim}{\zeta} \right\} d\tau$$

$$= - \int \left\{ \nabla \vec{\xi}^* : \left(\mu \nabla \frac{\partial \vec{\xi}}{\partial t} + \mu \vec{v} \cdot \nabla \nabla \vec{\xi} \right)_{sym} - \vec{\xi} \cdot \nabla \cdot \left(\nabla \underset{\sim}{\zeta} \cdot \vec{\xi}^* \right) + \right.$$

$$\left. \vec{\xi}^* \cdot \nabla \vec{\xi} \cdot \nabla \cdot \underset{\sim}{\zeta} + \frac{\mu'}{\mu} \underset{\sim}{\zeta} : \nabla \vec{\xi}^* \right\} d\tau$$

The full equation is multiplied by $\vec{\xi}^*$, the solution of the adjoint operator M and integrated over all $\vec{r}_0$. The nonviscous equations lead to Hermitian operators that give real integrals. As a result the stability in time could be obtained by solving for the real roots of the quadratic equations on ω. The full viscous equations must be treated in a more extensive manner. The individual operators are evaluated via a generalization of

Green's theorum stated as follows:

$$\int_V \nabla P \, dv = \int_{\partial D} P \vec{n} \, ds$$

where P is an arbitrary tensor function. In the present example $\vec{\xi}$ is multiplied to an arbitrary tensor quantity and integrated over all $\vec{r}_o$. Integrating by parts and making use of the conservations principles gives:

$$\int_V \nabla \cdot \left(\mu \nabla \frac{\partial \vec{\xi}}{\partial t} \right) \cdot \vec{\xi}^* \, dv = \int_{\partial D} \left(\vec{\xi}^* \mu \cdot \nabla \frac{\partial \vec{\xi}}{\partial t} \right) \cdot \vec{n} \, ds - \int_V \nabla \vec{\xi}^* : \mu \nabla \frac{\partial \vec{\xi}}{\partial t} \, dv$$

For these results it is assumed that on the boundary the components of $\vec{\xi}$ normal to the surface is zero. For most instances this integral will then be equal to zero if the divergence theorum holds. Other integrals are:

$$\int_V \vec{\xi}^* \cdot \nabla \cdot \left[\frac{\partial -1}{a^2} \left(\vec{v} \nabla \vec{\xi} \vec{v} + \vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} \right) \frac{\vec{\xi}}{\mu} \right] d\tau = \int_{\partial D} \vec{n} \cdot \left[\vec{\xi}^* \frac{\gamma -1}{a^2} \left(\vec{v} \cdot \nabla \vec{\xi} \cdot \vec{v} \right. \right.$$
$$\left. \left. + \vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} \right) \vec{\xi} /\mu \right] ds - \int_V \nabla \vec{\xi}^* \cdot \left[\frac{\partial -1}{a^2} \left(v \cdot \nabla \vec{\xi} \cdot \vec{v} + \vec{v} \frac{\partial \vec{\xi}}{\partial t} \right) \vec{\xi} \right] d\tau$$

The pressure perturbation is integrated by parts to give:

$$\int_V \vec{\xi}^* \cdot \nabla \left[\vec{v} \cdot \nabla \vec{\xi} \cdot \vec{v} \frac{\partial -1}{a^2} P \right] + \vec{\xi}^* \cdot \nabla \left(\vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} P \frac{\gamma -1}{a^2} \right) d\tau = \int_V \nabla \cdot \left[\frac{\gamma -1}{a^2} P \left(\vec{v} \cdot \nabla \vec{\xi} \cdot \vec{v} \right. \right.$$
$$\left. \left. + \vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} \right) d\tau - \int \left[\vec{v} \cdot \nabla \vec{\xi} \cdot \vec{v} + \vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} \right] \frac{\gamma -1}{a^2} P \cdot \nabla \cdot \vec{\xi}^* \right. d\tau$$

But the total divergence over the volume is zero because $\vec{\xi} \cdot \vec{n}$ is assumed to be zero on the boundary. Therefore, only the last term in the integral expression remains. Similarly:

$$\int_V \vec{\xi}^* \cdot \nabla (P \, div \vec{\xi}) \, d\tau = - \int_V P \, | \, div \vec{\xi} |^2 \, d\tau$$

The first viscous term is:

$$\int \vec{\xi}^* \cdot \nabla \cdot \left[\frac{1}{a_c} \left(\vec{V} \cdot \nabla \vec{\xi} \cdot \vec{V} + \vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} \right) \xi \mu \right] d\tau = \int_{\partial D} \left[\xi^* \frac{1}{a_c} (\vec{V} \cdot \nabla \vec{\xi} \cdot \vec{v} + \vec{v} \cdot \frac{\partial \vec{\xi}}{\partial t} \right] \xi \cdot \vec{n} \, ds - \int_v \frac{1}{a_c} \left[\vec{V} \cdot \nabla \vec{\xi} \cdot \vec{V} \cdot \frac{\partial \vec{\xi}}{\partial t} \right] \nabla \vec{\xi} : \tau \, d\tau$$

The second viscous term is given by:

$$\int \vec{\xi}^* \cdot \nabla \cdot \left[\mu \vec{V} \cdot \nabla \nabla \vec{\xi} \right] d\tau = \int_{\partial D} \vec{\xi}^* \mu \vec{V} \cdot \nabla \nabla \vec{\xi} \cdot \vec{n} \, ds - \int_v \mu \nabla \vec{\xi} : (\vec{V} \cdot \nabla \nabla \vec{\xi}) d\tau$$

Finally the integral involving the double inner product must be evaluated on a term by term basis.

The displacement energy must be real, but the operator that defines the shear terms is in general not symmetric and therefore not self-adjoint. The stability limit must then be obtained by finding the real part of the shear operator and adding it to the energy difference obtained from the inviscid equations to find the stability criterion. This is what was done by Sattinger[3] in showing and verifying that the stability results of Serrin[50] were true for more general problems. The use of the solution of the adjoint equation was necessary because this allows using the mathematics developed for the Fredholm alternative in Hilbert space. To obtain an approximation to the eigenfunctions w of the integral equation determined from the above system.

The general problem need not be solved and only real displacements are necessary to determine a solution. This simplifies the mathematical approach greatly, and allows using the real part of the displacement operator equation without restriction. The individual integrals within the shear operator may be integrated to determine the adjoint equation.

Thus:

$$\int_v \left(\vec{\xi}^* L \vec{\xi} - \vec{\xi} M \vec{\xi}^* \right) dv = \int_s \left(\vec{\xi}^* B \vec{\xi} - \vec{\xi} B' \vec{\xi}^* \right) ds$$

The boundary conditions are such that the surface integral goes to zero.

The first term under the integral is equal to the trace of $i\omega \nabla \vec{\xi} \cdot \nabla \xi_{sym}$ an imaginary quantity. The second is equal to:

$$\mu \, \xi^*_{\alpha,\beta} \, v_\ell \, \xi_{\alpha,\beta\ell} \big|_{sym} = \mu \, \xi^*_{\alpha,\beta} \, v_\ell \, \xi_{\alpha,\beta\ell} \big|_{sym}$$

The above has terms of the form $\vec{\xi}^* \cdot \dfrac{\partial \vec{\xi}}{\partial X} = \left(\vec{\xi}_c - i\, \vec{\xi}_i \right) \left(\dfrac{\partial \xi_r}{\partial x} + i\, \dfrac{\partial \xi_i}{\partial x} \right)$

$$= \vec{\xi}_r \, \frac{\partial \vec{\xi}_r}{\partial x} + i \left(\xi_r \frac{\partial \xi_i}{\partial x} - \xi_i \frac{\partial \xi_r}{\partial x} \right) + \xi_i \frac{\partial \xi_i}{\partial x}$$

But ξ_α, β $\tau_{\alpha\beta_0}$, ℓ ξ_ℓ, where $\nabla \vec{L} \cdot \vec{\xi}^*$ is a symmetric tensor. Therefore, only the symmetric part of tensors remain:

$$\vec{\xi}_{11} t_{11} + 2\, \xi_{12} t_{21} + 2\, \xi_{13} t_{31} + \xi_{33} t_{33} + \xi_{22} t_{22} + 2\, \xi_{23} t_{32}$$

The term $\vec{\xi}^* \cdot \nabla \vec{\xi} \cdot \nabla \cdot \vec{L}$ can be written as:

$$\vec{\xi}^* \cdot \nabla \vec{\xi} \cdot (\nabla \cdot \vec{L}) = \nabla \vec{\xi} \cdot (\nabla \cdot \xi) \cdot \vec{\xi}^* = \nabla \vec{\xi} : (\nabla \cdot \zeta) \vec{\xi}^*$$

The integral of this term is $-\int_v \xi \cdot \nabla \cdot \left[(\nabla \cdot \vec{L}) \vec{\xi}^* \right] dv$. The surface integrals are given as boundary conditions so that they go to zero.

This shows that all of the terms in the shear stress perturbations are either imaginary or complex. Combining all the terms that compose the stability equation results in a quadratic equation on ω with complex coefficients. The pressure perturbation and the fluctuating component of the coefficient of viscosity can have terms that depend on ω. It has been shown that the equation on ω is then:

$$-\omega^2 \int \rho \, |\xi|^2 \, dv + \omega \left[\int \vec{\xi}^* \cdot \rho \vec{v} \cdot \nabla \frac{\partial \vec{\xi}}{\partial t} \, dv + \frac{i}{R} \int \mu \, \nabla \vec{\xi}^* : \nabla \vec{\xi} \big|_{sym} \, dv \right.$$

$$\left. + \frac{\gamma-1}{R} \, i \int_v (\vec{v} \cdot \vec{\xi})(\nabla \vec{\xi}^* \cdot \xi) \, dv + (\gamma-1) i \int \rho \vec{v} \cdot \vec{\xi} \, \nabla \cdot \vec{\xi}^* \, dv + C_1 + i \, C_2 \right] = 0$$

$$A_1 = \int_V \rho |\vec{\xi}|^2 \, dv \qquad B_1 = \mathcal{R} \int \vec{\xi}^* \cdot \rho \vec{v} \cdot \nabla \frac{\partial \vec{\xi}}{\partial t} \, dv$$

$$B_2 = \frac{i}{\mathcal{R}} \int_V \mu \, \nabla\vec{\xi}^* : \nabla\vec{\xi} \Big|_{sym} \, dv + \frac{(\gamma-1)}{\mathcal{R}} i \int_V (\nabla \cdot \vec{\xi})(\nabla\vec{\xi}^* : \vec{\xi}) \, dv$$

$$+ (\gamma-1) i \int \rho(\vec{v} \cdot \vec{\xi}) \nabla \cdot \vec{\xi}^* \, dv$$

The solution for w is given by:

$$\omega = \frac{B_1 + i B_2}{2A_1} \pm \frac{1}{2A_1} \sqrt{B_1^2 - B_2^2 + 2 B_2 B_1 i - 4A(C_1 + iC_2)}$$

The real and imaginary parts of the above solution must be evaluated

because the form of the time dependent part of the displacement vector is

given by $e^{i\omega t}$. Thus, if the imaginary part of the solution is substituted

into the time dependent exponential, it becomes:

$$e^{-(\omega_I + i\omega_R)t} = e^{-\omega_I t} e^{i\omega_R t}$$

Since time goes from zero to t, the stability depends on the condition that

$w_I \geq 0$.

$$\frac{B_2}{2A} \pm \frac{1}{2A} \sqrt{r} \cos \theta/2 \geq 0$$

$$\frac{B_2}{2A} \geq \frac{1}{2A} \sqrt{r} \cos \theta/2$$

The term r is the absolute value of the term in the square root. The

angle θ is a measure of the value the imaginary part of the term within the

square root. If $\vec{\xi}$ is continuously differentiable in V and has only bounded

derivatives there, then the operator S is defined on all pairs $\vec{\xi}^*$ and $\vec{\xi}$ such

that:

$$|S(\vec{\xi}, \vec{\xi}^*)| \leq C_1 \int_V \nabla\vec{\xi} : \nabla\vec{\xi}^* \, dv \int_V \nabla\vec{\xi}^* : \nabla\vec{\xi} \, dv$$

and

$$\mathcal{R}_e \, S(\vec{\xi}, \vec{\xi}^*) \geq \int_V \nabla\vec{\xi} : \nabla\vec{\xi} \, dv - \int_V \vec{\xi}^* \cdot \vec{\xi} \, dv$$

The stability equation and its adjoint are defined, but because the operator on $\vec{\xi}$ is not symmetric, more than the first eigenfunction of S must be defined to form a basis in the Hilbert space. The adjoint is used because it forms an orthonormal subspace to the one defined by S.

Then for a sufficiently large number of eigenvalues, the real value of the above equation defines the energy of the system. The inviscid terms are real as was shown before. The real part of the shear terms are retained. This leaves the result:

$$-\omega^2 \int \rho |\vec{\xi}|^2 dv + 2\omega Re \int i\rho \vec{\xi} \cdot \vec{v} \cdot \nabla \vec{\xi} \, dv + C_{\kappa} = 0$$

Solving for ω gives:

$$\omega = \frac{2 Re\, i \int_v \rho \vec{\xi}^* \cdot \vec{v} \cdot \nabla \vec{\xi} \, dv}{2 \int_v \rho |\vec{\xi}|^2 dv} + \frac{1}{2 \int_v \rho |\vec{\xi}|^2 dv} \left\{ 4 \left(Re\, i \int \rho \vec{\xi}^* \cdot (\vec{v} \cdot \nabla \vec{\xi}) dv \right)^2 + 4 C_{\kappa} \int \rho |\vec{\xi}|^2 dv \right\}^{\frac{1}{2}}$$

The condition for stability implies that the term within the square root is greater than zero. An application using a simplified form of the above equation is discussed in the discussion in chapter 4.

CHAPTER 4. DISCUSSION

In the preceding two chapters the stability of an inviscid, non-heat conducting primary flow was examined. In chapter two the approach is one of dividing all the properties of the flow field into a steady state component and a fluctuctuating component with zero mean. This problem was further simplified by assuming that the steady state properties were functions of the radial coordinate only. This simplification made it possible to derive a stability equation for infinitesimal disturbances that was reduced to finding the conditions for which the amplitude of the disturbance could not grow in time. By either a Fourier transformation in time or the assumption of a normal mode, the stability equation became an ordinary differential equation in the radial coordinate with variable coefficients. Since there is no known general solution to this differential equation, some other means must be employed to determine the behavior of the solution in time. This was done by multiplying the equation by the complex conjugate of the solution and integrating overall r to eliminate the dependence on the space quantities. The stability criterion is then obtained by finding the condition for which the amplitude of the disturbance cannot grow. This is a standard approach for problems of this type and has been used since the earliest development of the stability of inviscid flows.

In chapter three configurational stability of a flow fluid is considered. This is done by assuming that the steady state properties may be

characterized to first order by assuming the value a flow property depends
on is the displacement relative to the equilibrium trajectory. This dis-
placement is both time and space dependent. The problem formulated in this
manner becomes Lagrangian representation of stability. By picking an arbi-
trary element that was displaced at time $t = t_o$, stability is determined
by considering whether the element returns to the equilibrium trajectory or
not. Again the stability problem reduces to an equation for which the spa-
cial dependence must be eliminated to determine a quadratic equation in the
frequency. If it can be shown that the roots of the equation are real, the
amplification factor is less than or equal to zero.

The criteria developed by either of the two methods is very complex.
Even in the simpler results of chapter two, there are too many parameters
to be able to state conclusively that a certain combination of one or more
flow properties determines stability in all cases. A discussion of the
general problem will follow, but it is possible to extract some significant
information by considering very simplified examples. One of these is a
straight axisymmetric jet. This problem has been studied in detail over
the years. The earliest recorded analysis of the jet is given in a paper
by Rayleigh dated 1892 in which he developed the equation for an incompres-
sible jet. In later papers he developed additional theories on the behavior
of jet flow including the concept of mode of maximum instability that allowed
prediction of the actual manner in which the system tended to break up.

The development of Rayleigh was followed in the paper by Batchelor and
Gill [20] in which they considered the effect that the velocity profile has
on the stability. They showed that axial velocity profiles that had a

maximum or minimum in the slope for a value other than at $r = 0$ are inherently unstable to axisymmetric disturbances. As an example, ring-shaped vortices appear at the boundary of the core very close to the orifice where the velocity is nearly constant then rapidly decreases to zero. These ring-shaped vortices were observed by Wehrmann and Wille and presented at a symposium in 1958. The existence of these ring-shaped vortices in liquid into liquid jet have been observed for years prior to their paper[21].

It is possible to apply the criterion developed for inviscid compressible flow in chapter two to determine if in the limit it produces the same results that were obtained in the classical papers mentioned. But it is also possible to examine the stability of very idealized jets that have been examined in the classical papers to see if simplified criteria can be obtained that will reveal some information for more difficult problems. Such problems include experiments on liquid into liquid jets and analytical calculations for shear layers and discontinuities of the velocity and density. The earliest experimental works for liquid-liquid jet were made by Savart[5] and analyzed later by Plateau[6] who concluded that the stability of the flow depends on the product of the radius of the jet and the wave number. Lord Rayleigh analyzed the problem later in his "On the Stability of Jets" and concluded that the system breaks up in its mode of greatest instability. By assuming that the jet has a finite radius it is possible to determine departures from its equilibrium state by a small perturbation approach. Chandrasekhar considered both gravitational and capillary instabilities in his book and concluded that a cylinder of fluid is unstable in a gravitational field for all wavelengths exceeding $\lambda = 2\pi R'x^*$ where x^* is the root

of $I_o(x^*)K_o(x^*) = \frac{1}{2} \Rightarrow x^* = 1.0668$, and is stable for all purely nonaxisymmetric disturbances.

For a jet held together by capillary forces the exterior force that has replaced the gravitational acceleration is the surface tension. Again the conclusion that the jet is stable to all purely nonaxisymmetric disturbances is obtained. A later work by Batchelor and Gill approached the problem from a different point of view. They considered a cylindrical vortex sheet for which both the inner and outer flow could be represented by a potential function. The solution they did obtain was for the low frequency limit because terms that would have included the second derivative with respect to time and the cross term of time and space were ignored. They did, however, arrive at a result that was similar to the one obtained by Chandrasekhar. The numerical result was different because there was no gravitational potential. Their condition becomes:

$$\left(\frac{u-c}{c}\right)^2 = \frac{K_n(\alpha a)}{k_n(\alpha a)} \frac{I'_n(\alpha a)}{I_n(\alpha a)}$$

The conlcusion was that the flow is unstable to small disturbances for all values of n and αa. In the limit as $\alpha a \rightarrow 0$ the two dimensional vortex sheet solution is recovered. For slow disturbances $\alpha a \rightarrow 0$, $(U - C)^2/C^2 \rightarrow 1$ for $n \neq 0$ and to zero for $n = 0$. Their conclusion of no changes in stability if the jet velocity gradient takes place over a finite thickness, d. Then disturbance $2\pi/\alpha \ll d$ are stable. If in addition $d/a \ll 1$, the two dimensional results should apply. The maximum growth rate of disturbances occurs when $\alpha d \approx 1$.

Since there is no gravitation or surface tension as in the work of Chandrasekhar, the choices for developing a compressible approach depend on

whether the method of Batchelor and Gill is retained or whether the analysis provided in chapter two is used. In the first case the linear perturbation equation comes directly from the generalized potential equation:

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \phi)^2 + \frac{1}{2} \nabla \phi \cdot \nabla (\nabla \phi) - a^2 \nabla^2 \phi = 0$$

It is assumed that the total potential is the result of a steady state plus a fluctuating component, the above equation may be reduced to the form with $\omega/\alpha a = n$ and $u/a = M$:

$$\phi' \left[(n+M)^2 \alpha^2 - \alpha^2 - \ell^2/r^2 \right] + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} = 0$$

This equation is valid for both the inner and outer flow. If the vortex sheet is assumed to be a surface of zero thickness the boundary conditions across this layer reduce to the continuity of the normal velocity and the pressure. At the center of the cylinder the perturbation solution must be finite and at infinity the Sommerfeld radiation condition must be satisfied. This assumption implies that disturbances only travel in the outward direction in the external flow field.

The perturbation equation has different solutions depending on whether the term $(n + M)^2 - 1$ is greater than or less than zero. This means that the effective Mach number defined by $n + M$ is subsonic or supersonic. For subsonic flows, the solution reduces to the one derived by Batchelor and Gill. Thus, except for the actual functional relationship that the wave speed and amplification factor may assume, the results are identical.

If the flow field is supersonic on one region or the other, then the solution in that region changes to a Bessel function of the first kind. These are oscillatory in the argument. Hence the condition that $C_i = 0$ is not uniformly satisfied for all possible combinations of the argument.

For instance if the inner flow is supersonic and the outer is subsonic, the inner solution is given by

$$\varphi_L = A \, J_\ell \left(\alpha r \sqrt{(n^2 + M)_L^2 - 1} \right)$$

and the outer solution is

$$\phi_o = B \, K_\ell \left(\alpha r \sqrt{(n + M)^2 - 1} \right)$$

Let the surface displacement be given by $\eta = E \, e^{i(\omega t + x\alpha + \ell\theta)}$.

The boundary conditions at the surface become:

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \Big|_{r=r_S} = \frac{\partial \phi}{\partial r} \Big|_{r=r_S} \qquad \text{and}$$

$$\rho \left(\frac{\partial \phi}{\partial t} + u_i \frac{\partial \phi}{\partial x_v} \right) \Big|_{r=r_S} = \rho_i \left(\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} \right)_o \Big|_{r=r_o}$$

The assumption of $M \gg n$ results in the condition that

$$\frac{(u_i - n)^2}{(u_o - n)^2} = \frac{K_\ell \left(\alpha a \sqrt{M_o^2 - 1} \right) J_\ell' \left(\alpha a \sqrt{M_i^2 - 1} \right)}{K_\ell' \left(\alpha a \sqrt{M_o^2 - 1} \right) J_\ell \left(\alpha a \sqrt{M_i^2 - 1} \right)}$$

The value of n can be real only if the right-hand side of the above expression is positive or zero. Since K_ℓ / K_ℓ' is less than zero for real arguments, the necessary condition for instability is that $J'_\ell / J_\ell > 0$. This is true only for certain ranges of the argument. If the axisymmetric disturbances are considered, i.e., $\ell = 0$, the expression becomes $J'_o / J_o = -J_1 / J_o$. The values between which the expression is negative depends on the zeroes of the functions J_o and J_1. Thus stability is achieved for arguments whose values lie between:

$$\alpha_{1,m-1} \leq \alpha r \sqrt{M_i^2 - 1} \leq \alpha_{o,m} \qquad m = 1, 2, \cdots$$

The above formulation would be altered if there were a stabilizing or destabilizing force because it would alter the zero value of the function. Thus, as the result decays to zero, it would either fall above or below this

new line, as is shown in the sketch:
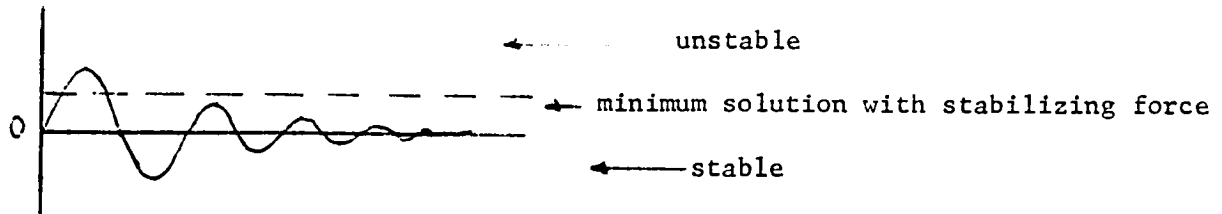
In the above problem the stability condition being oscillatory is pro-bably due to the geometry, because disturbances may travel through the cylin-der and act as either amplifying or cancelling waves. For the subsonic results the inner flow allows only disturbances that grow rather than oscil-late, thus they will always have the same sign and thus be destabilizing.

Rayleigh[4] was able to develop the linear stability equation by apply-ing a linearized perturbation approach to the equations of motion for inviscid, incompressible, unidirectional, axisymmetric flow. This equation was later rederived by Batchelor and Gill[20], and was given as a starting point for their analysis of the straight jet.

$$(u-c) \frac{d}{dr}\left(\frac{r}{\ell^2+\alpha^2 r^2}\frac{d(r q)}{dr}\right) - (u-c) q - r q \frac{d}{dr}\left(\frac{r u'}{\ell^2+\alpha^2 r^2}\right) = 0 \quad (4\text{-}1)$$

They showed that the stability of the jet with no external forces de-pended on the behavior of the term $Q'(r) = \frac{d}{dr}\left(\frac{r u'}{\ell^2 + \alpha^2 r^2}\right)$. This was accom-plished by multiplying the above equation by the complex conjugate of G and operating on the equation so that only the expression $C_i \ Q'(r)$ remained. This expression was equal to zero. The condition for stability in time requires that the imaginary part of C be equal to zero. The only condition that guarantees this result is that $Q'(r)$ is not zero anywhere in the flow field.

Rayleigh considered several simple examples that were based on the velocity profile:

$$U = A + Br^2 + D \ln r$$

For this profile the stability condition reduces to evaluating the derivative of $Q(r)$ given by:

$$Q'(r) = \frac{2Br^2 + D}{\ell^- + \alpha^2 r^2}$$

The function $Q(r)$ is monotonic in r and the derivative, $Q'(r)$, has the same sign for $r \to 0$. For axisymmetric flow with no inner tube or plug, i.e. $D = 0$, Q' is zero throughout the region of interest and nothing may be said concerning stability without solving the actual stability equation itself. The latter case comes from analyzing Poiseuille by this method.

Other conditions may also be derived. If a jet is assumed to have a top hat profile, it is easy to see that the slope of the profile is initially near zero, reaches a maximum at some $r \neq 0$ and again goes to zero as $r \to \infty$. In this case Q has a maximum somehwere in the flow field; and therefore, Q' must have a zero for a critical value of r. The stability equation as stated is then violated. The flow field is then subject to instability in the axisymmetric mode. This type of instability has been found explicitly in the form of ring-shaped vorticies near the orifice of a jet where the velocity is nearly constant.

If the velocity profiles vary only slowly with respect to r, it is required to examine the nonaxisymmetric disturbances for a source of the instability.

Howard and Gupta[30] showed that if the particular jet experiences instability, the amplification factor and the wave speed cannot have arbitrary values but are restricted to lie with the range:

$$[c_r - \tfrac{1}{2}(a+b)]^2 + c_i^2 \leq \tfrac{1}{4}(a-b)^2$$

where $a \leq U(r) \leq b$.

The compressible jet with rotation requires a similar approach. In either case the starting point is the differential stability equation, but because the coefficient $r/(1^2 + \alpha^2 r^2)$ has been replaced by a term that is complex, the actual technique that was used by Batchelor and Gill cannot be applied directly. An integral method is used that was developed in chapter two.

Now consider the compressible flow problem as outlined in chapter two. The general equation of motion is given by:

$$D\left(\frac{1}{\rho A}D^+\psi\right) - \psi/\rho + \frac{1}{\rho}\left\{\frac{L}{2}D_*\left(\frac{DL}{\rho A}\right) + \frac{(DL)^2}{4AL} + \frac{L}{A}\left(M\frac{D\alpha^2}{\alpha} + 3\left(\frac{\Omega}{\alpha}\right)^2 + D\ln\psi'\right)\right.$$

$$+ \frac{\ell}{AL}\left(M^2\alpha^2 - \Phi\right) + \frac{2\ell M\alpha^2 B}{AL^2} + \frac{2\ell^2}{AL}B + \frac{1}{\rho}D_*\left(\frac{\rho DL}{A}\right) + \frac{2}{\rho}D_*\left(\frac{\rho \Omega\ell}{rA}\right)$$

$$\left. - LD_*\left(\frac{1}{A}\right)(D\ln\rho - M)\right\}\psi = 0$$

Let $C = \frac{DL}{2} + \frac{2\Omega\ell}{r}$, then the above equation reduces to:

$$D\left(\frac{D^+\psi}{\rho A}\right) - \frac{\psi}{\rho} + \frac{1}{\rho}\left\{\frac{L}{A}\left[2M\,D\ln(r\alpha) + D^+D\ln\rho\right] - D\left(\frac{1}{A}\right)\left[LD\ln\rho\right.\right.$$

$$\left. - LM - C\right] + \frac{C}{A}D\ln\rho + \frac{1}{A}D_*C + \frac{C^2}{AL} + \frac{\ell}{AL}\left(M^2\alpha^2 - \Phi\right) + \frac{2M\alpha^2 B\ell}{AL^2}\right\}\psi = 0$$

Finally:

$$D\left(\frac{D^+\psi}{\rho A}\right) - \psi/\rho + \left\{\frac{ML}{r\alpha^2}D\left(\frac{r\alpha^2}{A}\right) + \frac{1}{\rho}D_*\left(\frac{C\rho}{A}\right) + \frac{1}{A}D^+(LD\ln\rho)\right.$$

$$\left. - LD\left(\frac{1}{A}\right)D\ln\rho + \frac{C^2}{AL} + \frac{\ell}{AL}\left(M^2\alpha^2 - \Phi\right) + \frac{2M\alpha^2 B\ell}{AL^2}\right\}\psi/\rho = 0$$

If the primary flow is an axisymmetric straight flow with no rotation, the above equation reduces to C = DL/2; M = 0; and $\bar{\Phi}$ = 0. The stability equation for the straight jet becomes:

$$D\left(\frac{D^*\Psi}{\rho A}\right) - \Psi/\rho + \frac{\Psi}{\rho}\left\{\frac{1}{\rho}D_*\left(\frac{C^2}{A}\right) + \frac{1}{A}D^+(LD_{h,\rho}) - LD_{h,\rho}D\left(\frac{1}{A}\right) + \frac{C^2}{AL}\right\} = 0 \quad (4-2)$$

The operators L and A are complex; therefore, it is required to write equation (4-2) by eliminating the imaginary parts from the denominator of the expressions, thus forming the terms $|A|^2$ and $|L|^2$ which are positive regardless of the choice of the frequency in the denominator. The equation is then multiplied by the complex conjugate of the solution Ψ^* and integrated over r to eliminate the spacial dependence:

$$\int -\left[\frac{\bar{A}}{|A|^2}|D^*\Psi|^2 + |\Psi|^2/\rho\right]r\,dr + \int \frac{1}{\rho}\left\{D_*\left(\frac{\rho C^2}{|A|^2}\right) + \frac{\bar{A}}{|A|^2}D(LD_{h,\rho}) - LD_{h,\rho}D\left(\frac{\bar{A}}{|A|^2}\right)\right.$$

$$\left. C^2\bar{A}\bar{L}/|AL|^2\right\}|\Psi|^2 r\,dr = 0$$

The operators may be expanded into real and imaginary parts:

$$A = A_R + i A_i = \frac{L_R^2 - L_i^2}{a^2} - \epsilon + \frac{2L_R L_i}{a^2}i \quad \text{and} \quad |A|^2 = A_R^2 + A_i^2$$

The imaginary part of the integral thus becomes:

$$\int \frac{A_i}{|A|^2}|D_*\Psi|^2 r\,dr + \int\left[\frac{1}{\rho}D_*\left(\frac{\rho C A_i}{|A|^2}\right) - \frac{A_i}{|A|^2}D^*(L_R D_{h,\rho})\right.$$

$$\frac{A_R L_I}{|A|^2}D^+D_{h,\rho} + L_R D_{h,\rho}D\left(\frac{A_i}{|A|^2}\right) - L_I D_{h,\rho}D\frac{A_i}{|A|^2} - c^2(A_i L_R + A_R L_I)/$$

$$|AL|^2\right]|\Psi|^2 r\,dr = 0$$

Dividing the entire expression by L_R/a^2, the first integral is a positive definite quantity and the second integral can vary depending on the particular steady state flow and the wave number of the disturbance. The

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amplification factor is zero if it can be shown that L_I is zero. Since L_I is the multiplier of the entire expression, this condition reduces to show that the integrand of the second integral does not change sign anywhere in the flow field.

$$\frac{a^2}{L_r}\left[-\frac{2}{\rho}D_n\left(\frac{c\rho L_r}{a^2|A|^2}\right) - \frac{2L_r}{a^2|A|^2}D^+(L_r D L_\rho) + \frac{A_r}{|A|^2}D^+(D L_\rho)\right.$$

$$\left. +2L_r D L_\rho \, D\left(\frac{L_r}{a^2|A|^2}\right) - D L_\rho \, D\left(\frac{A_n}{|A|^2}\right) - c^2\left(\frac{3L_r^2}{a^2} - \frac{L_r^2}{a^2} - \epsilon\right)\right]/|A L|^2 \geq 0$$

As can be readily seen both the density gradient and the speed of sound play important roles in this criterion. Since $|A|^2$ is positive definite, it plays no role in the actual stability criterion other than to change the magnitude of the various terms. Only in the terms that have A_R as a multiplier does L_I appear. Because of the form of the above expression there is a possibility that the equation may have more than one root. This condition exists if

$$L_I^2 - F_1/F_2 = 0 \tag{4-3}$$

where F_1 is all the quantities that do not have L_I multiplier and is all the quantities that are multiplied by L_I^2. If F_1/F_2 is less than zero, then there are no real values of L_I, and the only solution is the trivial one.

$$L_I \int \frac{|\frac{1}{A}|^2}{\left[\frac{1}{\rho}D_n\left(\frac{2\rho c L_r}{a^2|A|^2}\right) - \frac{2L_r}{a^2|A|^2}D^+(L_r D L_\rho) + \frac{A_r}{|A|^2}D^+(D L_\rho)\right.}$$

$$\left. +2L_r D L_\rho D\left(\frac{L_r}{a^2|A|^2}\right) - D L_\rho D\left(\frac{A_n}{|A|^2}\right) - c^2\left(\frac{3L_r^2}{a^2} - \frac{L_r^2}{a^2} - \epsilon\right)\right]/|A L|^2 \geq 0$$

The above integrand must be less than zero or the flow may have components that grow in time. There is still a possibility that there may be a zero in the integrand. This is true because L_I^2 appears in the integrand with a sign opposite to L_R^2. This gives an equation that is in the

form:

$$L_I^2 - F_1/F_2 = 0$$

with solutions given by $L_I = \pm\sqrt{F_1/F_2}$. If F_1/F_2 is positive, the solutions

for L_I are real. Thus, the stability criterion is a dual condition. The

two coefficients F_1 and F_2 are given by:

$$F_1 = \tfrac{1}{\rho}D_x \left(\frac{2\rho c L_k}{a^2|A|^2}\right) - D_r \left[\frac{(L_k^2 + c a^2)\, D h_\rho}{a^2|A|^2}\right] - \frac{c^2}{|AL|^2}\left(\frac{3L_k^2}{a^2} - e\right)$$

$$F_2 = D_x \left(\frac{2\rho c L_k}{a^2|A|^2}\right) - \frac{c^2}{a^2|AL|^2}$$

Thus the two conditions are $F_1 - F_2 L_I^2 < 0$ and $F_1/F_2 < 0$ for stability.

Since L_I^2 is positive or zero, the stability condition is then:

$$F_1/F_2 \leqslant 0$$

$$\frac{\frac{1}{\rho}D_x \left(\frac{2\rho c L_k}{a^2|A|^2}\right) - D_x \left[\frac{(L_k^2 + e a^2)}{a^2|A|^2}\, D h_\rho\right] - \frac{c^2}{|AL|^2}\left(\frac{3L_k^2}{a^2} - e\right)}{D_r \left(\frac{D h_\rho}{a^2|A|^2}\right) - \frac{c^2}{a^2|AL|^2}} \leqslant 0 \qquad (4\text{-}4)$$

The addition of gravity to the equation of motion gives a perturbation

potential through the term $\rho'g$. Depending on which momentum equation con-

tains the gravitational body force, effects the stability of the entire

system. If the gravity vector is aligned with the r direction, the gravity

vector and any centrifugal force are interchangeable. In equation (4-4),

there are several terms composed of the sum of squares, $\dfrac{c^2 e}{a^2|AL|^2}$ and $\dfrac{c^2}{|AL|^2 a^2}$.

For incompressible flow, the condition for instability as given by Batchelor

and Gill is that the expression C e has a turning point in the flow field.

This condition is satisfied for all velocity profiles that have a maximum

in the slope for r ≠ 0. For this condition the first term in the numerator

has a zero somewhere in the flow field, and therefore, yields the condition that the flow field may be unstable. For compressible flow with C^2 much greater than the remaining terms reduces to:

$$3L_R^2 / a^2 < e$$

This auxiliary condition is obviously satisfied if $L_R^2 \ll e$, but there is a speed at which the sign of the expression changes and instability is possible even if the incompressible flow were stable to the same perturbation. It has been shown that for supersonic flows disturbances that travel faster than the local speed tend to be destabilizing. This is true of compression waves that steepen as they propagate. For a jet it has been shown experimentally that disturbances that are stable in form for incompressible flow may in fact act as destabilizing ones supersonically. [55]

Examining the full equation for rotation and axial flow the condition for stability is given by:

$$\left\{ \frac{2}{\rho} D_r \left(\frac{\rho c L_R}{a^2 |A|^2} \right) - D_r \left(\frac{L_R^2 - L_I^2 - e a^2}{a^2 |A|^2} D_r \rho \right) - \left(\frac{3L_R^2}{a^2} - \frac{L_I^2}{a^2} - e \right) \left(c^2 - e \Phi - e M^2 a^2 \right) \Big/ |AL|^2 \right.$$

$$\left. + D_r \left[\frac{M(L_R^2 - L_I^2 - e a^2)}{a^2 |A|^2} \right] + 2 L_R D_r \left(\frac{M L_R}{a^2 |A|^2} \right) - \left(\frac{L_R^2 + L_I^2 + e a^2}{a^2 r^2} \right) D(\Omega^2 r^4) \Big/ |A|^2 \right. \quad (4\text{-}5)$$

$$\left. + \frac{4 B c M L_R}{|AL^2|^2} \left(L_R^2 - L_I^2 - e a^2 \right) \right\} < 0$$

Again it is possible to define dual conditions on the stability of the flow because of the appearance of L_I^2 with a sign that is opposite to L_R^2, the above equation also reduces to one with the form:

$$F_1 - F_2 L_I^2 < 0$$

where F_1 is the condition if it is assumed that $L_I^2 = 0$. Therefore, stability can occur only if two conditions are met. These are given by evaluating

the coefficients F_1 and F_2 and giving the limits as follows:

$$F_1 = \frac{2}{3} D_\star \left(\frac{\rho C L_\star}{c^2 |A|^2}\right) - D_\star \left[\frac{(L_\star^2 + \epsilon c a^2)(D L_\star \rho_o - M)}{a^2 |A|^2}\right] - \frac{1}{|AL|^2}\left(\frac{3 L_\star^2}{a^2} - \epsilon\right)(\qquad (4\text{-}6)$$

$$c^2 - \epsilon \Phi + \epsilon M^2 a^2) + \frac{4\epsilon B M L_\star}{|A L^2|^2}(2 L_\star - \epsilon a^2) - \frac{L_\star^2 + \epsilon a^2}{a^2 r^3 |A|^2} D(\Omega^2 r^4) < 0$$

$$F_2 = - D_\star \frac{(D L_\star \rho - M)}{a^2 |A|^2} + \frac{1}{a^2 |AL|^2}(c^2 - \Phi e + \epsilon M^2 a^2) \qquad (4\text{-}7)$$

$$- \frac{D \Omega^2 r^4}{a^4 r^3 |A|^2} - \frac{8 \epsilon M B L_\star}{|AL|^2} \leq 0$$

In the above expressions the term $(\ln \rho_o - M)$ appears. It is one of the components of the Brunt-Väisälä frequency that determines the static stability of a compressible gas in a force field. Its first use occurred in a paper discussing the stability of the atmosphere by Väisälä in 1925. He developed an equation relating the displacement of a fluid element relative to its initial position and found that the buoyancy ratio appeared as a frequency in the equation of motion. The second order differential equation that he derived had either oscillatory or exponential solutions depending on whether this frequency was positive or negative. As a result he was able to establish the distribution of the density in the atmosphere for adiabatic conditions. For swirling flow the gravity vector is replaced by the centrifugal acceleration. Therefore, static stability here depends on the sign of the expression. In the present form the expression is not recognizable, but using the steady state normal momentum equation it can be cast into more easily recognized form:

$$D \ln \rho - M = \frac{D \rho}{\rho} - \frac{\Omega^2 r}{a^2} = \frac{D \rho}{\rho} - \frac{D P}{\rho a^2}$$

The corresponding frequency is then given by: $\left(\frac{D\rho}{\rho} - \frac{D\rho}{\rho a^2} \right) \Omega^2 r =$

$$\left(\frac{D\rho}{\rho} - M \right) M a^2 = a^2 M \left(\frac{D\rho}{\rho} - M \right) < 0 \qquad \text{for static stability.}$$

In the above equation the buoyancy term acts as a dynamically desta-
bilizing expression if the flow is assumed to be statically stable. But
like the atmosphere where the stable dry air results lead to the adiabatic
lapse rate, the sign of the derivative of the buoyancy is the critical para-
meter.

In deriving the above equations (4-6) and (4-7) it is determined that
some of the terms are the derivatives of positive quantities. The Richard-
son number criterion as originally developed implied that for a swirling
flow to be stable:

$$\Phi / c^2 > 1$$

This condition is satisfied in equation (4-6) via the third term. In
addition there are terms which depend on the behavior of the velocity profile.
The first term in (4-6) is the same as the one derived by Rayleigh[4] and
Batchelor and Gill[20] for incompressible flow. If all other effects are
negligible it states that the flow is stable or unstable depending on whether
the term $D_* \frac{\rho_0 C L_R}{a^2 |A|^2}$ has a zero in the flow field for $r \neq 0$ or that the ex-
pression is positive throughout the flow field. For a jet $x = \frac{\rho_0 L_R}{a^2 |A|^2}$ is a
positive number if there is no reverse flow.

Thus $D_*(xC) < 0 \Rightarrow rD(xC/r) < 0$. Stability is achieved only if xC/r
decreases outward from the center line. For a profile that has a zero C for
some $0 < r < r_C$ and again for $r > r_C$, the expression has a minimum or maximum

in the flow field. This yields a zero for the first term. The second term
is the derivative of a positive definite quantity (if the flow is statically
stable). Thus the only possibility of a stabilizing effect is for the term
to be growing faster than r. This is unlikely; therefore, the effect of a
statically stable buoyancy is to have a destabilizing dynamic effect. The
fourth term depends on both the velocity profile and the effective Mach num-
ber, L_R/a. Disturbances that are stabilizing at one effective Mach number
are destabilizing above the critical Mach number and vice versa.

The last term is stabilizing if the rotational velocity increases out-
ward from the center line and is basic to the theorem stated by Rayleigh.
The dual criterion is then first that:

$$\frac{|AL|^2 F_1 a^2 + (3L_R^2 - e a^2)(c^2 + e M^2 a^2)}{(3L_R^2 - e a^2)(c^2 + e M^2 a^2)} > 1 \qquad (4\text{-}8)$$

The second condition is that:

$$\left\{ a^2 |AL|^2 \left[D_\alpha \left(\frac{D \zeta_{\wp} - M}{a^2 |A|^2} \right) + \frac{1}{\alpha^2 r^2 |A|^2} D(\Omega^2 r^4) - \frac{g e \beta M L}{|AL|^2} \right. \right. \qquad (4\text{-}9)$$

$$\left. \left. + \frac{e \Phi}{a^2 |A|^2} \right\} \Big/ (c^2 + e M^2 a^2) \geq 1$$

The conditions state that in order for the flow to be stable (4-8) must
be satisfied. Stability is possible provided that the shear plus rotation
are sufficiently small so that the numerator is always greater than the de-
nominator. Excluded from the above are the conditions near the singular
point $\frac{3L_R^2}{a^2} = e$, because non-linear terms that were dropped are needed to
eliminate the singular nature of the problems. Equation (4-8) is not suf-
ficient to guarantee stability if (4-9) is not satisfied, because non-trivial
solutions will be possible given by $L_I = \pm\sqrt{F_1/F_2}$. The condition of no
rotation reduces the expression to the one derived earlier and simply states

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that the density gradient must increase in the direction of decreasing axial velocity. This is similar to the Richardson number criterion that states that:

$$\frac{1}{4}\left(\frac{du}{dz}\right)^2 < -\frac{g}{\rho}\frac{d\rho}{dz}$$

where g is in the negative z direction. This states that the density must increase more rapidly than the square of the velocity for stability. With no force field the condition becomes one on the density gradient and the velocity gradient alone.

The above results are new. Even to go this far greatly complicates the analytical process. We will now examine the proposed method given in Chap. 3. The method of perturbing the flow and thermodynamic variables, then eliminating components until only one remains does not allow considering problems that go much beyond the scope of one-dimensional examples. An actual steady flow satisfies both the equations of motion but also identically satisfies the stability conditions as well. If the stability conditions are not satisfied, the configuration of the flow will change so that it will approach a more stable condition. This type of stability is defined as configurational stability. This type of instability is exhibited by a physical change in the observed flow field. An example of such a flow is the appearance of circulation in the flow about a profile in a circulation-free stream.

The first part of chapter 3 follows the development of Burshtein and Solov'ev[42], and Frieman and Rotenberg[43]. In this method the steady state solution is assumed to be known throughout the entire space. These properties are then associated with a particular collection of elements that compose a fluid volume that travels on an equilibrium trajectory in

this space.   A particle, composed of identifiable fluid elements, is

displaced from its trajectory at an earlier time, t_o, and the forces acting

on it are derived to determine if the fluid volume will return to its tra-

jectory or follow a new path. The trajectory is defined by the vector $\vec{r}_o$

and the displaced trajectory by $\vec{r}_o + \vec{\xi}$. Since the fluid elements within the

volume are the same, the principle of conservation of mass is satisfied

identically. The incremental forces acting on the body are given to the

first order in terms of the displacement vector $\vec{\xi}$ by using the conservation

of mass principle and the remaining conservation equations. If the flow is

inviscid, the deformation of the fluid volume is the result of the pressure

and convective momentum gradients that are defined via the steady state

solution. If the fluid has viscosity the deformation of the fluid volume

must also include the rates of strains of the volume. Since the shape of

the fluid volume examined is arbitrary, the effects of the viscous forces

may be represented by the transformation of a local coordinate system defined

on the fluid volume as it is translated through the assumed force field. The

coordinate system will in general experience a rotation and a non-uniform

stretching. The definition of the strain tensor is given by the difference

of the metric tensors for the two local coordinate systems. We will assume

that there are no rotational discontinuities or angular momentum sources in

the fluid; therefore, by the conservation of angular momentum, the strain

tensor must be symmetric. The deformation tensor is composed of both a

rotational antisymmetric component plus the strain tensor, but only the

symmetric component is needed to determine the force due to the shear. The

time rate of change of the deformation times the coefficient of viscosity

gives the shear stress tensor.

In the equations of motion the resulting force that acts on the surface is the product of a material quantity (viscous coefficient) and the time rate of change of the deformation. These are surface integrals that describe the kinematic relationships. The coefficient of viscosity may also change because it depends on both the material and the associated thermodynamic quantities such as the temperature. Thus, when the surface integral is changed to a volume integral via the divergence theorem, more terms appear in the equation of motion. The fact that the divergence changes as a function of $\vec{\xi}$ can be demonstrated by using the continuity equation. It can also be demonstrated with a straight-forward application of the transformation matrix. The last term that involves the shear stress is obtained by eliminating the static pressure gradient. The energy equation for the fully viscous problem is not the isentropic relationship. It may be obtained via one of three approaches. The first is to assume that the total temperature is constant along streamlines. This results in equations that involve the Bernouli equation for compressible flow. The second is to assume that the pressure density relation may be approximated locally by letting the exponential γ^c go to a new number c to account for the entropy production. The final way is to write the second law of thermodynamics such that:

$$\rho \frac{DS}{Dt} + \nabla \cdot \vec{q} = \frac{1}{T} \underset{\sim}{\tau} : \nabla \vec{v} + \frac{k}{T^2} \nabla T \cdot \nabla T$$

In most cases, we are dealing with $\vec{q}$ assumed to be zero. Therefore, the entropy production is equal to dissipation energy which is a positive quantity. Since $\frac{DS}{Dt} = \frac{DP}{Dt} - \frac{1}{a^2} \frac{D\rho}{Dt}$, one or both of the terms is positive. Effectively, all of the viscous contribution to the equation of stability is composed of the difference in total shear force between the volume at

$\vec{r}$ or $\vec{r} + \vec{\xi}$. Depending on the type of a steady state the entire energy

addition is either positive, negative or zero. Conceptually, the viscous

expression should act as forcing functions that will increase or decrease

the value of the quantity $F(\vec{\xi})$ in the non-viscous expression. The energy

is a real quantity; therefore, as was shown by Sattinger[3] the real part

of the operator is retained in the stability analysis.

In the present analysis, the form of the eigenvalue equation is not

standard because of the appearance of terms multiplied by ω^2. If it is

assumed that ω is an eigenvalue of S, it is possible to write the equation

in an integral representation. The form of the equation then is:

$$f(x) = g(x) + \lambda \int K(x,r) \, g(r) \, dr + \lambda^2 \int K_2(x,r) \, f(r) \, dr$$

If $g(x) + \lambda \int K(x,r) \, g(r) \, dr$ belongs to L_2 and $K_2(x,r)$ belongs to L_2,

then the Fredholm theory in Hilbert space applies. This was shown to be the

case in the problem solved by Frieman and Rotenberg. Thus they were able

to write:

$$\mathcal{S}f = \lambda F$$

$$\mathcal{S}^\dagger g = \lambda^* g$$

and
$$\lambda = \frac{g \cdot \mathcal{S} \cdot f}{g^* \, f} \qquad \mathcal{S}\lambda = 0$$

The problem for which we may state that:

$$R_e \, \mathcal{S}(\vec{\xi}, \vec{\xi}^*) \geq \omega \, |\vec{\xi}|^2$$

The problem of determining the conditions for which there may be growing

solution of S requires solving the quadratic equation based on the integral

of the linearized perturbation equation. Then by defining $F'(\vec{\xi})$ as the

entire set of terms that do not explicitly depend on time, ω becomes:

$$\omega = \left(\int \rho |\xi|^2 d\tau \right)^{-1} \left\{ - \int i \rho \vec{\xi}^* \cdot (\vec{v} \cdot \nabla \vec{\xi}) d\tau \pm \left[(\int \rho \vec{\xi}^* \cdot (\vec{v} \cdot \nabla \vec{\xi}) d\tau)^2 \right. \right.$$
$$\left. \left. - \int \rho |\vec{\xi}|^2 d\tau \int \vec{\xi}^* \cdot F'(\xi) d\tau \right]^{1/2} \right\}$$

For those conditions for which all the surface integrals are eliminated by the homogenous boundary condition, a sufficient condition for the stability of a particular flow reduces to the condition the term under the integral is positive. The time dependent contribution resulting from the kinematic shear condition is imaginary and therefore drops from the equations of motion. By limiting the sufficiency condition to $-\int \vec{\xi} F' d\xi$ (a more severe restriction than the above) it is possible to consider only real perturbations.

The result may be integrated over an arbitrary path and time to form the action appropriate for the system. Since we are considering only real perturbations, the integral will be identical to that of Frieman and Rotenberg in form but the F' function is much more complicated due to the addition of the viscous and non-isentropic process results. Additionally, the real part of the time derivative which originates from the shear stress terms adds to the second term in the integral. But because the more restrictive condition given by $\int \vec{\xi}^* \cdot F(\vec{\xi}) d\tau \geq 0$ is used, the variational problem becomes:

$$\delta \omega_r = -\frac{1}{2} \int \vec{\xi}^* \cdot F(\vec{\xi}) d\tau$$

We will introduce a curvilinear coordinate system that is defined relative to stream surfaces about a central axial streamline. Then there will be only two velocity components $\vec{V} = (V_1, V_2, 0)$. The vector given by $\frac{\text{grad } \psi}{|\text{grad } \psi|}$ is in the normal coordinate direction. The problem may be solved by letting the displacement vector $\vec{\xi}$ be represented by a series of orthogonal eigenfunctions that define the basis for the Hilbert space.

The perturbation equation that was developed in chapter three and is
stated above, gives a method whereby combinations of steady state properties
may be examined to find the boundaries of stability to infinitesimal pertur-
bations. It is desired to find the growth or decay of these disturbances
in time. Therefore, since time does not appear explicitly in the resulting
equation, it is possible to assume a harmonic form for the time dependence
of the displacement vector. It is assumed to be separable and does not
enter into the actual computations. By writing it as $e^{i\omega t}$, stability reduces
to finding those conditions for which ω is real, or if complex for which
only decay is possible.

The perturbation equation itself gives the force acting on a fluid
element as it is displaced from its equilibrium trajectory by the vector, $\vec{\xi}$.
Since we are interested in the fate of the particle of fluid itself, the
equations are written following a particular fluid element. It can be
shown to the order of the approximation that the form of the equations in
either the Lagrangian or Eulerian representations are similar. The meaning
is different because a particular fluid element is followed in the present
problem. To the order of the approximations used the substantial derivative
is independent of the perturbation path. The individual terms within the
perturbation equation may be divided into those terms that make up the
second total derivative with respect to time, terms that represent the
changes in volume or density of the element, terms that depend upon the ener-
gy that is stored as a pressure, and those terms that represent losses to
the system in the form of entropy production or viscous effects. All of these
terms are given as forces that act on the named particle. These distur-
bance forces are multiplied by the displacement length and integrated over
a time interval or path length. Those terms that are derived from the sub-
stantial derivative become $\vec{\xi} * \frac{d^2\vec{\xi}}{dt^2} \Rightarrow |\frac{d\vec{\xi}}{dt}|^2$. If this quantity is multiplied
by the mass of the fluid element it becomes the kinetic energy of the per-
turbation motion. The remaining terms give the potential energy and the
viscous dissipation energy. The latter is a measure of the losses genera-
ted as a result of letting the fluid be both viscous and heat-conducting to
the order of the perturbation.

If the flowfield is inviscid, it was shown in chapter three that the operators that make up the equation of motion for the disturbed element are Hermitian. This means that the equation for ω has only real coefficients. The resulting quadratic equation may then be solved, and the condition for stability is reduced to finding those discriminants that are positive. If the viscous terms are included, then it is no longer possible to guarantee that the individual operators are self-adjoint. This means that it is possible to find flow fields for which a more involved analysis is required. The possibility exists that the terms from the viscous dissipation operator are all real with the exception of those terms that involve the partial derivative with respect to time explicitly. This is the same condition that is obtained in kinematics of particles subjected to both conservative forces and dissipative forces. In that example whether it is possible to diagonalize the system of equations or not, the dissipation function is always positive and the effect of the dissipation is always damping regardless of its magnitude. The only exceptions to this rule are those limited values of the disturbance frequency that coincide with the natural frequencies of the system. The arguments that were used to derive the motion of a particle or system of particles may be extended to continuous media by replacing the summations by integrals and by replacing the point functions by continuous functions on position.

In the case of a fluid motion the magnitude of the dissipative terms and the non-viscous destabilizing forces may be such that even though the effect of time contribution to the motion may be stabilizing, the discriminant introduces imaginary components of ω, and the flow field is destabilized by the addition of the viscosity.

The argument above allows one to obtain qualitative information about the stability of a general flowfield subject to a given disturbance. This may not be sufficient for the particular problem that is to be examined. For this reason it is often necessary to find criteria that are independent of the perturbation applied to see if such flows are unconditionally stable.

To derive such a condition for the general viscous flow is left to a future work. This paper will stop at an equation from which such an analysis may take place. It is possible to derive such information for a class of inviscid flow fields.

This problem reduces to a simpler form because of the Hermitian property of its operators. As stated earlier the sufficiency condition is derived by examining the discriminant of the quadratic equation on ω. The two parts that make up this expression are given as follows:

$$\left| \int \rho \, \vec{\xi}^* \cdot (\vec{v} \cdot \nabla \vec{\xi}) \, d\tau \right|^2 - \int \rho |\vec{\xi}|^2 d\tau \int \vec{\xi}^* \cdot F(\vec{\xi}) d\tau$$

The integral of $\vec{\xi}* \cdot F(\vec{\xi})$ is given after combining several terms to give a positive definite expression plus several other terms of undetermined sign.

$$\vec{\xi}^* \cdot F(\vec{\xi}) = CP \left| d_{iv} \vec{\xi} - \vec{\xi} \, \frac{\rho \vec{v} \cdot \nabla \vec{v}}{CP} \right|^2 - A |\vec{v} \cdot \nabla \vec{\xi}|^2 - \left(\frac{\rho \vec{v} \cdot \nabla \vec{v}}{CP} \right)^2 |\vec{\xi}|^2 - R_e \, \vec{\xi}^* \vec{\xi} : \nabla(\rho \vec{v} \cdot \nabla \vec{v})$$

It can be shown that the first term in the magnitude of the first integral above may be dropped in comparison to the second term in $\vec{\xi}* \cdot F(\vec{\xi})$. This is shown by use of the Schwartz inequality as follows:

$$\left| \int \rho \, \vec{\xi}^* \cdot (\vec{v} \cdot \nabla \vec{\xi}) \, d\tau \right|^2 \leq \int |\rho \, \vec{\xi}^* \cdot (\vec{v} \cdot \nabla \vec{\xi})|^2 d\tau \leq \int \rho |\xi|^2 \rho |\vec{v} \cdot \nabla \vec{\xi}|^2 d\tau$$

$$\leq \int \rho |\vec{\xi}|^2 d\tau \int \rho |\vec{v} \cdot \nabla \vec{\xi}|^2 d\tau$$

This shows that the stability condition abbreviates to

$$CP |d_{iv} \vec{\xi} - \vec{\xi} \, \frac{\rho \vec{v} \cdot \nabla \vec{v}}{CP}|^2 - K_1 \rho |\vec{v} \cdot \nabla \vec{\xi}|^2 - \left(\frac{\rho \vec{v} \cdot \nabla \vec{v}}{CP} \right)^2 |\vec{\xi}|^2 - R_e \, \vec{\xi}^* \vec{\xi} : \nabla(\rho \vec{v} \cdot \nabla \vec{v}) \geq 0$$

where $0 < K_1 < 1$

In the above expression, the displacement vector $\vec{\xi}$ appears both as a derivative and by itself. Unless these may be grouped such that $\vec{\xi}$ does not appear except as a multiplier, the criterion cannot be achieved. It is known that the above expression must be equal to or greater than zero for stability. It is also easy to see that several of the above terms are negative definite. Thus, a condition for minimum stability is derived by

letting the positive definite expression go to zero. The new condition is
then given by

$$\left[K_1 |\vec{v}\cdot\vec{\nabla}\vec{\xi}|^2 + \frac{\rho(\vec{v}\cdot\vec{\nabla}\vec{v})^2}{CP}|\vec{\xi}|^2 + \vec{R}_e \vec{\xi}\vec{\xi} : \vec{\nabla}(\rho\vec{v}\cdot\vec{\nabla}\vec{v})/\rho \leq c \right]$$

This gives a relationship between some of the derivatives of the dis-
placement vector and the displacement via the steady state quantities. This
term also demonstrates a very interesting property of a variable density
flow field. In the initial derivation of the equations of motion it was
shown that the volume of the fluid element and its density were related.
It was shown at that time that the div$\vec{\xi}$ was equivalent to the density per-
turbation that the fluid element undergoes during the motion from point a
to point b. For an inviscid fluid not in a force field the pressure force
and the inertial forces are in equilibrium. As a result the expression
$\rho(\vec{v}\cdot\vec{\nabla}\vec{v})\cdot\vec{\xi}$ is approximately equal to the rate of change of the pressure
force acting on the fluid element as it is displaced. If the pressure
field is assumed to act as a stationary force field associated with the
space, then the expression is a buoyancy force acting on the fluid element.
This is shown by considering that the pressure field acting on the displaced
particle remains equal to that of the undisturbed motion. When it is
assumed that there is a change in the pressure with position, the force
acting on the particle after translation is different from the one at its
equilibrium position. If the new force tends to return the parcel of
fluid to its original trajectory, the effect is to enhance stability. Thus,
stability implies that div$\vec{\xi} > \vec{\xi}\cdot\frac{\rho\vec{v}\cdot\vec{\nabla}\vec{\xi}}{CP}$. This is a side condition on stabi-
lity for compressible flows. Because in this derivation there could be a
small entropy production the ratio of the specific heats was not γ.

The first negative definite term may be expanded to give

$$|\vec{v}\cdot\vec{\nabla}\vec{\xi}|^2 = \frac{1}{4}|grad(\vec{v}\cdot\vec{\xi}) - \vec{v}\times curl\vec{\xi} - \vec{\xi}\times curl\vec{v} - curl(\vec{v}\times\vec{\xi}) + \vec{v}div\vec{\xi} - \vec{\xi}div\vec{v}|^2$$

Stability is an intrinsic property of the flow field. Therefore,
changing the reference coordinate system should not in any way alter the
outcome of the analysis. As a result it is possible to write the steady

state variables in a coordinate system that is related to the steady state
stream surfaces. In this way, the steady state solution depends primarily
on the curvature of the stream surfaces themselves. Let the stream surfaces
be defined relative to some reference streamline in the flowfield. Let
there be a single stream function such that $\text{grad}\psi \cdot \vec{v} = 0$. The components of
$\vec{v}$ are defined by the continuity equation. Thus

$$\frac{\partial}{\partial \xi_1}\left(h_2 h_3 \rho u\right) + \frac{\partial}{\partial \xi_2}\left(h_1 h_3 \rho v\right) = c$$

The continuity equation and the definition of a stream surface give the
condition of no mass flux normal to the surface. The displacement vector
is arbitrary, so for this example it is assumed to be in the direction of
$\text{grad}\psi \cdot \vec{v}$. For this condition only certain of the terms in $|\vec{v}\cdot\nabla\vec{\xi}|^2$ remain.

$$|\vec{\xi}|^2 \left[\left(\frac{u}{h_1 h_2}\frac{\partial h_1}{\partial \xi_3}\right)^2 + \left(\frac{v}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right)^2\right] + \left|\frac{u}{h_1}\frac{\partial \xi}{\partial \xi_1}\right|^2$$

The result is a criterion that depends primarily on the curvature of the
stream surface itself. This is true of the effective rate of change of
the pressure gradient in the direction of the displacement and in the vor-
ticity of the steady flow as defined by the curvature of the surface. The
derivative of $\vec{\xi}$ with respect to q_1 is the rotation of the displacement
vector. It is assumed to be of smaller magnitude than the other terms in
the expression.

By dividing the remaining rate of change of the pressure gradient by
the vortical terms squared, one obtains a criterion that is similar to the
Richardson number. In this equation the standard value of 1/4 has been
replaced by k_1, a number between zero and one.

$$\rho h_3 \frac{\partial}{\partial \xi_3}\left(\frac{\rho u^2}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3} - \frac{\rho v^2}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right)\Big/\left[\left(\frac{u}{h_1 h_3}\frac{\partial h_1}{\partial \xi_3}\right)^2 + \left(\frac{v}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right)^2 + \frac{2}{c_1 k_1}(\vec{v}\cdot\nabla\vec{v})^2 + k_2 u^2\right] > k_1$$

The above problem may be solved again, but this time the displacement
is in the plane of the stream surface but normal to the velocity vector.
Again the only undefined term comes from $\vec{v}\cdot\nabla\vec{\xi}$. For this example the coor-
dinates were chosen such that the velocity vector was in the q_1 direction,
$\vec{e}_2$ was in the direction of the displacement and $\vec{e}_3$ was normal to the surface.

Again because the stability must be an intrinsic problem of the motion, it is possible to derive the criterion

$$\frac{1}{\rho} \nabla(\rho \vec{v} \cdot \nabla \vec{v}) \Big/ \left\{ \left(\frac{1}{h_1}\frac{\partial v}{\partial \zeta_1}\right)^2 + \left(\frac{v}{h_1 h_2}\frac{\partial h_1}{\partial \zeta_2}\right)^2 + \left(\frac{v}{h_1 h_3}\frac{\partial h_1}{\partial \zeta_3}\right)^2 + \frac{\rho(\vec{v}\cdot\nabla\vec{v})}{c^2\rho^2 \kappa_1} \right\} > \kappa_1$$

The final condition that may be analyzed is to allow the displacement to be in the direction of the velocity vector. This condition becomes:

$$\frac{1}{\rho} \nabla\left(\frac{\rho v^2}{h_1 h_3}\frac{\partial h_1}{\partial \zeta_2}\right) \Big/ \left\{ \frac{1}{4}\frac{v^2}{h_2^2}\left(\frac{1}{h_1}\frac{\partial h_1}{\partial \zeta_2} + \frac{1}{h_3}\frac{\partial h_3}{\partial \zeta_2}\right)^2 + \frac{v^2}{(h_1 h_2)^2}\left(\frac{\partial h_2}{\partial \zeta_2}\right)^2 \right\} > \kappa_1$$

The Richardson number normally contains derivatives of the axial velocity normal to the direction of motion. This is in actuality the square of the circulation or vorticity. Thus, for a compressible flow, stability depends first on the direction of the buoyancy forces and secondly on the magnitude of the vorticity of the flow field. The first expression may be positive or negative depending on the various parameters of the problem, but the vorticity terms are always positive; therefore, they act to desta-bilize the flow.

The previous development of this criteria by other authors was res-tricted to the case in which the steady state properties were a function of the coordinate normal to the direction of motion only. Further, they were restricted by the type of geometry that could be used. The above develop-ment does not have these drawbacks. It is therefore, a generalization of the previously derived results. It also shows that the primary driver for instability is the vorticity of the fluid motion. If the flow field is to remain stable for all time, a mechanism must be present that will over-come the destabilizing effects of the circulation. The equations for irrotational flows must be examined using the basic equation as a starting point and a slightly modified analysis must be used. For rotational flow this can be a favorable centrifugal force field. For the stability of the atmosphere, the dual conditions of a gravity field and a favorable buoyancy force must be present. In general the stability of the types of flows that have been considered require that the force increment tending to return a fluid parcel to its equilibrium position be greater than the sum of the

effect of the vorticity percent in the steady state flow and pressure
gradient that accelerates the particles away from equilibrium.

The criteria defined above do not resemble those of other authors
because of the coordinate system chosen. By reverting to a more generally
used system, it can be shown that the metric derivatives are more complex
representations of velocity gradients. To get to the new system, a von
Mises type transformation was performed. Each coordinate is defined via
the solution of the Pfaffian differential equation $d\vec{s} \times \vec{v} = 0$. The contin-
uity equation has the solution given by $\operatorname{div}\rho\vec{v} = \operatorname{div}(\operatorname{curl}\vec{A}) = 0$. If a trans-
formation is made that cancels only the mass flow across a surface $\rho\vec{v} =$
$\rho\vec{v}_1 + \rho\vec{v}_2 = \rho\vec{v}_1 + \operatorname{curl}\vec{A}$. But the definition of a stream function is
$dw\mu(\operatorname{grad}\psi_1 \times \operatorname{grad}\psi_2) = 0$. Since this is $\operatorname{curl} A_1$, $\operatorname{grad}\psi_1 = \vec{e}_2$ and $\operatorname{curl}\vec{A}_1 =$
$\vec{e}_2 \times (\operatorname{grad}\psi_2\big|_{g_3}\vec{e}_3 + \operatorname{grad}\psi_2\big|_{g_1}\vec{e}_1)$. Now q_3 is the parameter in the solution
of the differential equation. The remaining coordinate q_1 and q_2 are
defined via the function ψ_2 and the Serret-Frenet equations on the surface.
They must have a common normal and be perpendicular to each other in the
tangent plane. Then the metrics h_1, h_2, and h_3 in the new coordinate
system can be defined relative to the old system and the definition of the
stream function.

$$\frac{\partial \dot{\psi}_3}{\partial \dot{y}_1} = -\rho \omega_2 \quad , \qquad \frac{\partial \dot{\psi}_3}{\partial \dot{y}_3} = \rho \omega_2$$

$$h_\nu = \sum_j^3 \left(\frac{\partial x}{\partial \dot{q}}\right)^2 = \sum_j^3 \left(\frac{\partial x}{\partial \dot{y}_i}\,\frac{\partial \dot{y}_i}{\partial \dot{y}}\right)^2$$

The reduced form of the metric is

$$h_\nu = f(h_\ell')/g(\rho u_2)$$

Then $\qquad \dfrac{\partial h_\nu}{\partial \dot{y}} = \dfrac{\partial h_\nu'/\rho u}{\partial \dot{y}} = \dfrac{\partial h'}{\partial \dot{y}}\,\dfrac{1}{\rho u} - \dfrac{h'}{\partial u^2}\,\dfrac{\partial \rho u}{\partial \dot{y}}$ $\qquad$ for the

simplest example. The end result is that both equations depend on the
velocity gradients. Hence, the forms are, in fact, similar.

Consider the $\vec{\xi}^{*} \cdot F'$ function given by

$$- \vec{\xi}^{*} \cdot F' = cP\, |div\, \vec{\xi}|^2 - \rho|\vec{v}\cdot\nabla\vec{\xi}|^2 - Re\,(2\rho\vec{v}\cdot\nabla\vec{v}\;\vec{\xi}^{*}div\,\vec{\xi})$$

$$Re\;\vec{\xi}^{*}\vec{\xi}\cdot\nabla(\rho\vec{v}\cdot\nabla\vec{v}) - Re\,\nabla\vec{\xi}^{*}:(\mu\vec{v}\cdot\nabla\nabla\vec{\xi})|_{sym}/R + Re\,[\vec{\xi}\cdot\nabla(\nabla\xi\cdot\vec{\xi})]/R$$

$$- Re\,\vec{\xi}^{*}\cdot(\nabla\vec{\xi}\cdot\nabla\underset{\sim}{\xi})/R - Re\,\frac{\mu'}{\mu}\underset{\sim}{\xi}:\nabla\vec{\xi}^{*}$$

For cylindrical symmetry it is possible to define the orthogonal functions as a Fourier series and to consider only a particular harmonic:

$$\vec{\xi} = \vec{\xi}_c(r)\,e^{i\ell\phi\,+i\alpha x}$$

Combining some of the above terms:

$$cP\,|div\,\vec{\xi}|^2 - 2Re\,\rho\vec{v}\cdot\nabla\vec{v}\;\vec{\xi}^{*}div\,\vec{\xi} = cP\left[div\,\vec{\xi} - \vec{\xi}\,\frac{\rho\vec{v}\cdot\nabla\vec{v}}{cP}\right]^2$$

$$-\left(\frac{\rho\vec{v}\cdot\nabla\vec{v}}{cP}\right)^2 |\xi_i|^2$$

The minimum value of this expression is given when:

$$div\,\xi = \frac{\rho\vec{v}\cdot\nabla\vec{v}}{cP}\cdot\vec{\xi}$$

Only the normal pressure is balanced. Thus:

$$div\,\underset{\sim}{\xi} = \frac{\rho\Omega^2 r}{cP}\,\xi_r$$

Another term is derived from the last inviscid expression:

$$\xi_r^2\,\frac{\partial}{\partial r}\,(\rho\Omega^2 r)$$

The viscous forces act to retard the motion and balance the pressure forces in the normal direction only via the terms that indicate a departure from the cylindrical geometry, such as $-\frac{2u}{r}\frac{\partial v}{\partial\theta}$. In the example of cylindrical flow these terms are zero.

The convective terms that do not express themselves via the curvature $(\vec{V}\cdot\nabla V\theta,\ \vec{V}\cdot\nabla V_x)$ are zero in the anlaysis. If the components of the velocity

and the thermodynamic quantities depend on both r and x then the additional

terms $V_1 \frac{\partial V\theta}{\partial X}$ and $V_1 \frac{\partial V_x}{\partial X}$ must be included in the equation for stability.

The shear stress tensor has but four terms $\tau_{32} = \tau_{23}$ and $\tau_{13} = \tau_{31}$. These

are a measure of the velocity gradients in the direction normal to the stream

surfaces. In order to evaluate the terms that involve the expression $\xi^* \frac{\partial \xi}{\partial X_i}$

we use the integral equality:

$$\int f^2 dx = -2 \int x \, f \, f' \, dx \qquad\qquad \text{then}$$

$$\int \dot{x}^2 f'^2 dx \int f^2 dx \geqslant \left(\int x \, f \, f' \, dx \right)^2 = \tfrac{1}{4}\left(\int f^2 dx \right)^2$$

The most important of the shear terms are then:

$$\tfrac{1}{4} |\vec{\xi}|^2 (\tau_{13} + \tau_{23}) / \text{Reynolds no} + |\vec{\xi}|^2 \nabla^2 (\tau_{13} + \tau_{23}) / \text{Reynolds no.}$$

+ positive kinematic shear terms

The resulting expression for stability is then given by:

$$\int r \, dr \left\{ r \tfrac{\partial}{\partial r} (\rho \Omega^2) - \frac{\rho \Omega^4 r^2}{c_\rho} - \tfrac{1}{4r} (\tau_{23} / \text{Reynolds no}) + \nabla^2 \tau_{23} / \text{Reynolds} \right\} |\xi|^2 \geqslant 0$$

or

$$\tfrac{\partial}{\partial r} (\rho \Omega^2) - \frac{\rho^2 \Omega^4 r}{c_\rho} - \tfrac{1}{4r} (\tau_{23} / \text{Reynolds}) + \nabla^2 \tau_{23} / r \text{ Reynolds} + KS \geqslant 0$$

The above is true for the case of closed streamlines and a small gap

between layers. This allows one to make the expansions required to reduce

the gradients of $\vec{\xi}$ into an expression that involved only products of $\vec{\xi}$. The

alternate approach which is equally valid if $\vec{\xi}$ and its derivatives to the

third order all remain finite in the space over which the integration takes

place is to replace the integral of the derivatives by a constant times

the integral over the volume. The value of the constant is not arbitrary

but must meet criterion such that if:

$$(Tf, g) = \ell_g \, f \qquad \text{then} \qquad |\ell_g (f)| \leq \| Tf \| \, \| g \| \leq c \| f \|$$

where
$$C = \|T\| \cdot \|\mathfrak{g}\|$$

For the rotational problem the shear stresses tend to slow the motion of the fluid particles away from the trajectory. Therefore, they can be stabilizing. This is shown by the fact that the kinematic shear terms are shown to increase the magnitude of the positive terms in the stability criterion. The non-viscous solution reduces to the Rayleigh-Synge condition that the angular momentum has to increase for increasing radius. The stability of the rotating flow to small perturbations is the result of the pressure gradient $\frac{\rho V_\theta^2}{r}$ that is present to return a displaced particle to its initial state. For viscous flow the shear stress and the pressure both tend to slow the motion of a displaced particle. This means that the flow field will be slightly more stable for viscous flow than for non-viscous flow.

The compressible viscous flow problem has several shear perturbations that do have the same sign as the angular momentum gradient. These will tend to act to destabilize the motion of a particle leaving the trajectory.

Another problem that may be solved easily using this method is that of a compressible fluid in a temperature field. If there is no motion initially i.e. $\vec{V} \equiv 0$, the problem reduces for an isentropic process to

$$\int s P \, |d\omega \xi|^2 = \int \vec{\xi}^* \cdot F' d\tau \geq 0$$

This term is the potential energy density as obtained by resorting to a Hamiltonian formulation. Taking a variation with respect to the energy yields the potential energy part of the Lagrangian density functions. In order for the oscillations or the displacements to yield stable results

the potential energy must have an absolute minimum for arbitrary displacements $\vec{\xi}$. This is shown to be true because the variation of the integral is the potential energy, equal to zero yields the minimum value of a quantity that is positive definite.

There are more difficult problems that can be solved using the method that is indicated but they will not be given here. The obvious extension that this method allows is also including the dependence of quantities in directions other than the radial one for problems such as the swirling jet. This is because in the final form of the stability equations all the derivatives of the steady state variables occur as gradients. If one is examining the motion of a particle of fluid for which the dependence of the properties in two directions are needed, the form of the simple problem of rotational flow remains unaltered. The stability will depend on the relationship of the velocity V^{ϕ} in both the r and X directions for the more complicated problem . The other velocity component V_X enters via the term $V_X \frac{\partial \xi}{\partial X}$. For these problems the functional relationship of $\frac{\partial \vec{\xi}}{\partial X}$ must be known to determine whether a flow is stable or not. For this problem the additional term to be considered is:

$$\rho \left| V \cdot \nabla \xi \right|^2 = \rho \left| V_x \frac{\partial \xi}{\partial x} \right|^2$$

In addition the expression for the convective term is:

$$\rho^2 \frac{|V \cdot \nabla V|^2}{\gamma \rho} |\xi|^2 = \frac{\rho}{a^2} \left[\left(V_x \frac{\partial V_x}{\partial x} \right) |\xi_r|^2 - \frac{V_\phi^4}{r^2} |\xi_r|^2 + V_\lambda \frac{\partial V_\phi}{\partial x} |\xi_\phi|^2 \right]$$

with corresponding changes in the viscous shear term. Hence:

$$- \left[\frac{\partial}{\partial r} \left(\frac{\rho V_\phi^2}{r^2} \right) + \frac{\rho^2 V_\phi^2}{\gamma \rho r^3} \right] |\xi_r|^2 - \frac{\rho^2 |\xi_r|^2}{\gamma \rho} \left| V_x \frac{\partial V_x}{\partial x} \right|^2 - \frac{\rho^2}{\gamma r} \left| V_x \frac{\partial V_\phi}{\partial x} \right|^2 |\xi_\phi|^2 \geq c$$

This shows the flow field is more unstable if the flow field has components that depend on X for cylindrical geometries.

The above development does not produce a parameter such as the Richardson number because in order to obtain this parameter, the equations must be reduced by serially eliminating the individual components of the perturbation quantities. The definition of constant mass elements eliminates some of the interdependence of components that were used to make up the ratio. The Richardson number for fluid flows with swirl is a measure of the ratio of the induced pressure gradient and buoyancy to the vorticity of the flow field as given by the square of the velocity gradient. In the present analysis, it is impossible to develop the last term without trying to reduce the equations to one that depends on one of the components of $\vec{\xi}$ only. Doing this would destroy the simplicity of the method and add little to the information derived.

The need for a method that will compute the stability of a flow field that depends both on the radial and axial direction, is exemplified if the velocity components are examined. The velocity components may be written in functional form as:

$$u = f(x) \, g(\eta) \qquad\qquad \text{fig. (4a, 4b)}$$

$$w = f_1(x) \, g_1(\eta) \qquad\qquad \text{fig. (5)}$$

where:

$$\eta = \frac{r}{x}$$

where a_1 is the location of the effective origin of the flow. Let the swirl number S be given by the formula:

$$\int_o^\infty \rho u w \, r^2 \, dr \Big/ \int_o^\infty (\rho u^2 + P) \, r \, dr$$

This number is the ratio of the axial flux of angular momentum to the total head due to the axial flow which is the sum of the axial momentum flux and the pressure at a given station. As the swirl number increases from 0 to .6, the radial spread of the jet increases and the maximum centerline velocity decreases until the axial adverse pressure gradient exceeds the magnitude of the forward kinetic forces, ρu^2, and a recirculating region forms on the axis. This is accomplished by a reduction in the length of the potential core, a displacement of the maximum velocity to an off-axis location at $S = .5$ and finally reverse flow above $S = .6$. Above this critical value of S, large shears and intense turbulent mixing take place. This is the process that is desired for a swirl combustor. For intermediate swirl rates the velocity gradients are reduced. The pressure gradient is favorable with regard to the buoyancy forces and the flow is stabilized. This stabilization is the same as is obtained in the density stratified atmosphere and is due to the fact that more dense fluid elements will fall and lighter ones rise in an increasing "gravity field". If the density is inverted with respect to the force field, the system is unstable.

Most jets diverge after they leave the exit due to the actions of the pressure and viscous forces. This introduces a significant axial velocity gradient in all the flow parameters of the jet. For a flow exiting a divergent nozzle, the radial growth of the jet is enhanced. Therefore, the axial gradients are even greater. The decay rate of the swirl velocity of the jet with swirl imposed in the nozzle is twice as fast as the decay rate of the axial velocity. Thus, the importance of the terms depending on X derived from the terms of the expression $\vec{\xi} \vec{\xi}^* : \nabla(\rho \vec{V} \cdot \nabla \vec{V})$ can diminish in the problem.

The quantity $\rho\vec{V} \cdot \nabla\vec{V}$ may be represented by a vector $\vec{a}$ that represents the equations of motion. The symmetric part of the tensor $\nabla\vec{a}$ is retained. If the second derivatives with respect to the coordinate directions are ordered in magnitude, then certain of the terms of $\nabla\vec{a}$ that contain derivatives with respect to the axial direction are included. These will have the opposite sign that $\frac{d}{dr}(\rho w^2/r^2)$ has and thus contribute to destabilize the flow. This is in fact what happens with any flow that has a positive axial pressure gradient. Thus, the result is more general than the current application.

The manner in which the stability criterion was applied was to try to eliminate the gradient of $\vec{\xi}$, the perturbation displacement, by applying it locally over a small interval. This results in global considerations that are applicable for stability in time as long as the approximation employed is applicable everywhere. The alternative is to determine a $\vec{\xi}$ and do the actual integration over all applicable regions of configuration space. Alternatives are to use the Schwartz inequality relationships for integrals to eliminate the terms that are unimportant and obtain a somewhat more restrictive criterion. These applications will be left to a later work.

CONCLUSIONS

The objective of this report was to determine the conditions for which a flowfield in equilibrium would be stable to infinitesimal disturbances. Two analytical methods were chosen. In chapter two, the classical method of perturbing the equations of motion was used to determine an equation that could be analyzed to determine the assymtotic behavior of its solution in time. It was determined that the solutions which do not grow in time are stable. The second was to examine the forces that are acting on a particular mass of a fluid that was initially contained within the boundary $S(t_o)$. The mass was to remain within this surface for all time. It was also assumed that all the steady state properties of the fluid were functions of the position only. The behavior of the fluid element was examined as it was displaced from its equilibrium trajectory by the perturbation displacement, $\vec{\xi}$. A number of interesting simplifications are possible because the conservation of mass principle is satisfied identically. For compressible flows, it is possible to relate the change in the density to the elastic change in the volume of the element as it moves in time. Through the integral of the energy equation, the perturbation pressure and the perturbation density are related to the Jacobian of the transformation of the element from its equilibrium system to its displaced system. By retaining terms only to order $\vec{\xi}$, it is possible to write the equations in a form that is similar whether the coordinate system of the perturbation is relative to the equilibrium trajectory or the displaced trajectory. The resulting equation gives the forces acting on the parcel of fluid as a function of time and position in space. The form of the equation is in the Lagrangian format because the solution of the problem will yield the position of the fluid element as a function of time and all the properties of the fluid element are then derived via the equations that relate the steady state and the displaced state.

Having determined the form of the equations of motion in the perturbation sense, it is shown that it is not always necessary to solve the differential equation to determine whether the flow-field in question is stable or not. For a restricted set of problems it is possible to show that the stability depends only on the steady state properties. Several such problems were examined using both methods. These were primarily attempts to show that the classical results could be determined. There are several drawbacks to either method, the classical one is the result of perturbations of the thermodynamic quantities and the velocity. For a fully compressible problem there may be as many as seven dependent variables that must be eliminated in a serial manner before the basic differential equation may be found. This elimination proves most difficult except for those problems for which the linearized equations are represented by flow properties that depend only on one of the coordinate directions. The perturbation quantities must be represented as either a harmonic in a form that allows elimination by integral transforms. Any other formats would require the use of digital means. This greatly limits the applicability of the method because each new set of conditions would require a new solution to the entire problem. The problems that could be solved are primarily those classical problems that exist in the literature. This method of course can be extended if a suitable transformation can be found that reduces the problem to a more apparent form.

The problems that were examined using this method were the compressible rotating jet and the compressible cylindrical shear layer. The results of the computations were compared with the results of Fung and Kurzweg in the density stratified limit for rotating flow in finite domains. It was shown that a Richardson number type of criterion was possible. The stability was shown to depend on the ratio of the density and centrifugal force gradients divided by the square of the shear terms squared. This of course was the solution that was obtained by Fung and Kurzweg. In the case of fully compressible flow it is possible for the density to change as the fluid is displaced. As a result it is possible to have an additional effect

due to the buoyancy of the fluid. Each of the equations is solved by multiplying the perturbation equation by a perturbation velocity component. The result is akin to the mechanical energy balance of the fluid. Lallas was able to derive a transformation using a particular part of the energy integral to reduce the complicated fully compressible result to one that resembled the old density stratified criteria except that it contained the Brunt-Väisäilä frequency as an integral part of the criteria. In the present analysis the assumptions were slightly different from those of Lallas and no simple transformation could be found that was as effective as Lallas's in reducing the final form of the criteria. As a result, the present analysis showed that the Burnt-Väisäilä must be positive independently of the Richardson number. There were also several additional terms that involved the gradient of the operator L in the criteria. The two act in the same manner as the single stability statement of Lallas. In either case the qualitative result is that the ratio of the angular momentum gradient and the square of the rotational velocity derivative must be greater than $\frac{1}{4}$ for stability. This shows that a central force field such as is induced by the rotation can increase the inviscid stability of a flow field provided that the buoyancy force tends to restore displaced particles to their equilibrium positions and the force is large enough to overcome the action of the vorticity.

The basic compressible equation as derived here could also be reduced in the limit to the Rayleigh problem of a shear layer. The point of inflection theorem was based on these results. Lord Rayleigh was able to show that the appearance of an inflection point in the gradient of the axial velocity was sufficient condition to give instability. Because the equations were sufficiently alike, it was possible to show that this was the case for compressible flow also.

The stability of a cylindrical vortex sheet was determined by assuming that the Sommerfeld radiation principle applied. The inner and outer media were assumed to be uniform but with different properties. Unlike the incompressible result shown by Batchelor and Gill, who were able to show

that the solution was based on Bessel functions with real arguments.
The current solution consisted of a series of Hermite polynomials with
complex arguments. As a result stability was determined to be a function
of the frequency of the disturbance. The stability depended on which range
of the polynomial were examined. The character of the solution depended
also on whether the inner and outer flow were both supersonic, subsonic, or
one of each. It was determined that under the conditions of the problem
the stable solutions for fully supersonic flow were excluded. It was not
determined whether this was a shortcoming of the analysis or the equation
of motion.

The results derived in chapter two have several important shortcomings.
The most significant is that the properties of the flow field may not vary
in the axial direction. This is a valid condition to impose if the flow
field is confined in a straight pipe, but if the flow is in an unbounded
medium, the axial velocity gradients are very important. It was shown by
Chigier that imposing a rotational component from the outside of the jet
using a screen does indeed increase the stability of the jet edge. It is
also known that naturally imposed rotation will decay much more rapidly
than the axial velocity; therefore, it is important to see if these gradients
will in fact affect the overall stability of the flow. It is impossible to
do so using the method outlined in chapter two. Therefore, the configura-
tional stability approach was extended in chapter three to provide a means
of answering these questions. This method was developed to find some of
the stability problems associated with plasma physics. It was adapted
to be used for fluid mechanics to be able to extend the analytical results
to more general problems. As was stated earlier, the basic method is based
on a Lagrangian approach to the problem. The equations of motion as derived
are multiplied by the conjugate of the displacement vector and integrated
over a suitable space. A more rigorous mathematical analysis would use the
adjoint of the displacement vector as a multiplier. The result is that the
above gives the mechanical energy of the volume over which the integration
takes place. The inviscid convective terms yield the kinetic energy, the

the surface integral gives the work done by the shear forces acting on the surface, the pressure times the divergence of the elastic energy and the remaining shear terms give the viscous dissipation. For the inviscid flow problem, it was possible to obtain several criteria that resembled the Richardson number for rotational flow. These could be balanced such that the potentially stabilizing forces were listed in the numerator of the expression and the negative definite quantities in the denominator. As was expected there are only a limited number of situations for which the stability is entirely free of the frequency of the applied disturbance. But the form of the condition as given does not differentiate between a flow field that depends on more than one independent variable. Even for the more general cases as outlined in chapter four, it is still the vortical nature of the equilibrium flow that gives the mechanism for instability.

If the flow is either density stratified or incompressible, then the energy equation as given does not apply. Therefore, it is necessary to use the curl of the above equation to eliminate the arbitrary pressure perturbation. Even though the form of the equation is not greatly changed the resulting equation is much more difficult to integrate. As a result, the simple two dimensional problems such as the Kelvin-Helmhotz instability are more complex. It is not possible to divorce the form of the perturbation of the stability criterion. If it is possible to use the simplest example of a shear layer in a gravitational field, the stability solution depends on the wave number in the axial direction for two-dimensional flow. This result can be derived as well as the inflection point criteria without too much difficulty and was left out of the paper.

The result is that a method was developed which is especially suitable for compressible flows. This method gives a vector equation that relates the force on the fluid parcel to the perturbation displacement vector. As a result the basic differential equation depends only on the components of the displacement vector. This in itself is a great simplification. The

conditions for incompressible or density stratified flow are much more complex because the curl of the perturbation equation is used to eliminate the arbitrary perturbation pressure. The resulting form is much more difficult to integrate to get meaningful results.

The full equation for compressibly viscous force with laminar shear and small departures from isentropic flow was developed using the Lagrangian approach. This method can yield stability in two manners even if it cannot be shown that the equation on the frequency, ω, is real. If it can be shown as in the case of a particle in kinematics that the sign of the imaginary term in the shear stress components is positive, then the viscous terms would give an exponential decay with time. Under these circumstances the magnitude of any imaginary part resulting from the discriminant of the equation must be examined closely to determine if the conditions of stability are met. A more rigorous variational principle might be a desirable extension of this method.

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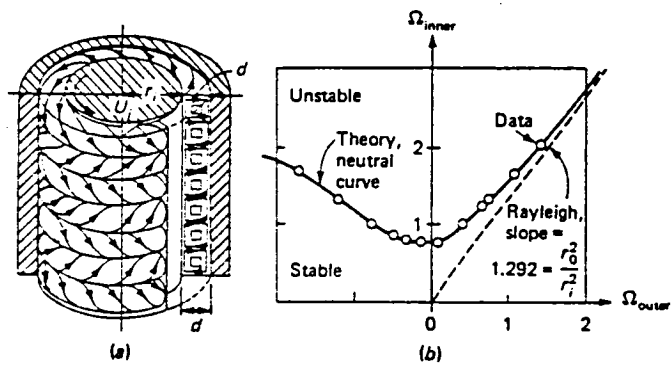
Fig. 1   Theory and experiment by Taylor (1923) for the instability of Couette flow
between rotating cylinders: (a) Taylor vortices; (b) theory and experiment for
$r_o = 4.035$ cm, $r_i = 3.55$ cm.

EFFECT OF SWIRL ON STABILITY
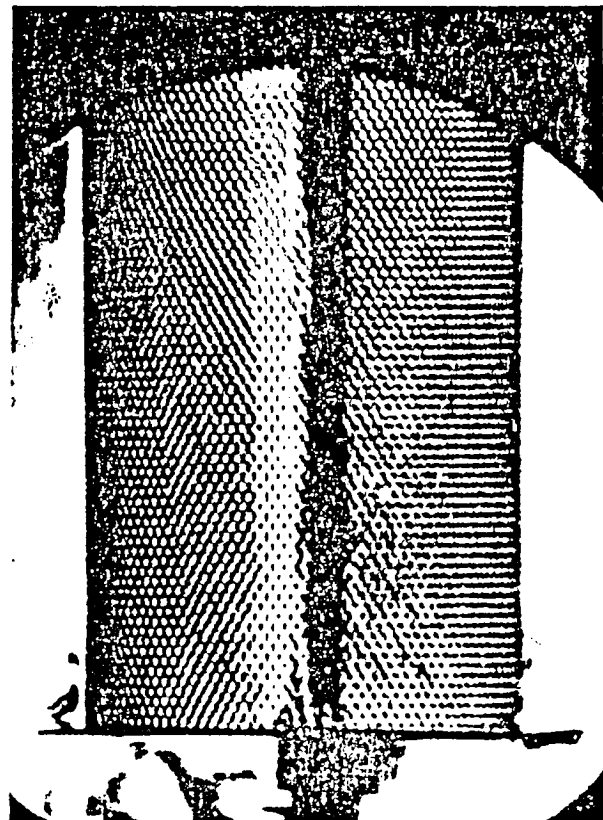
a) STATIONARY SCREEN   b) ROTATING SCREEN

FIG. 2 Laminarization of a turbulent jet flame by using a wire screen to rotate surrounding air. Schlieren photographs. (a) stationary screen, (b) rotating screen
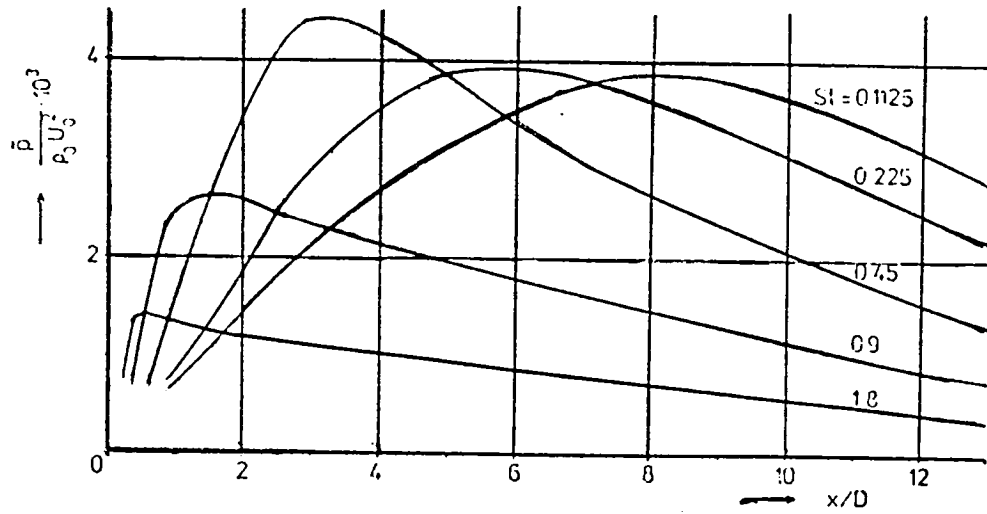
Fig. 3a Variation along the jet of unresolved frequency
components of the pressure, r = D/2.
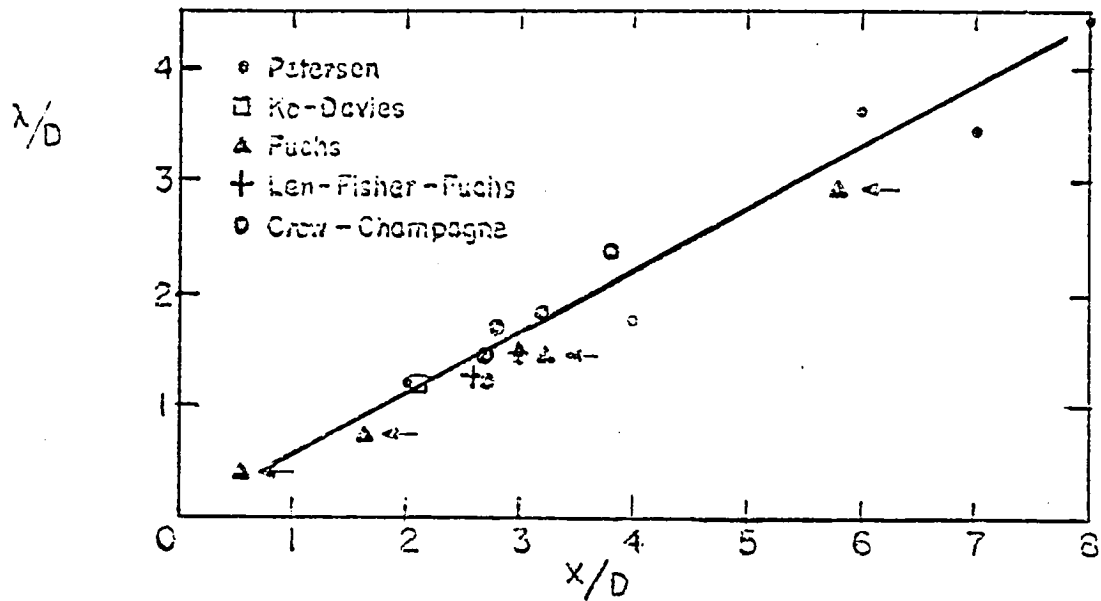
Fig. 3b Mean spaaration distance between neighbouring large-
scale structures in a jet after LAUFER et al.[29]
Arrows indicate the results from an evaluation of
extremes in Fig. 1a by using U_c/U_c = 0.65.
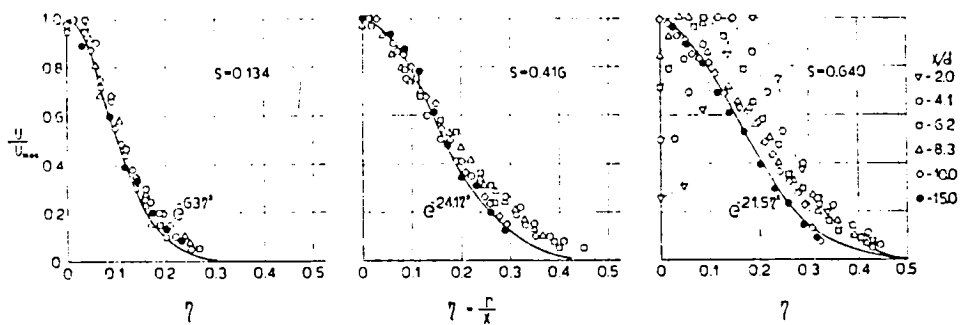
Fig. 4a. Radial distribution of axial velocity.
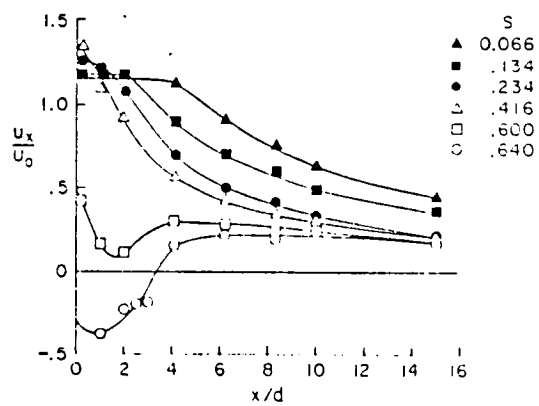
Fig. 4b. Axial component of velocity on axis of round
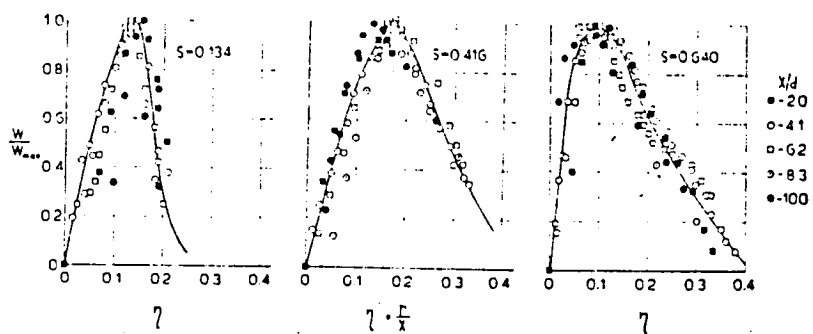swirling jet as function of axial distance and
swirl number (Ref. 19).

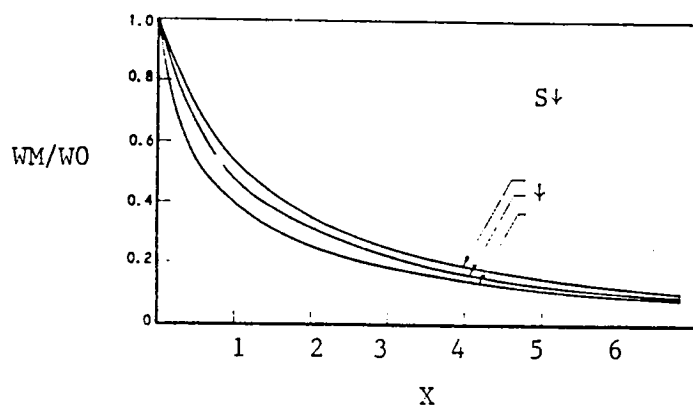Fig. 5a. Radial distribution of swirl
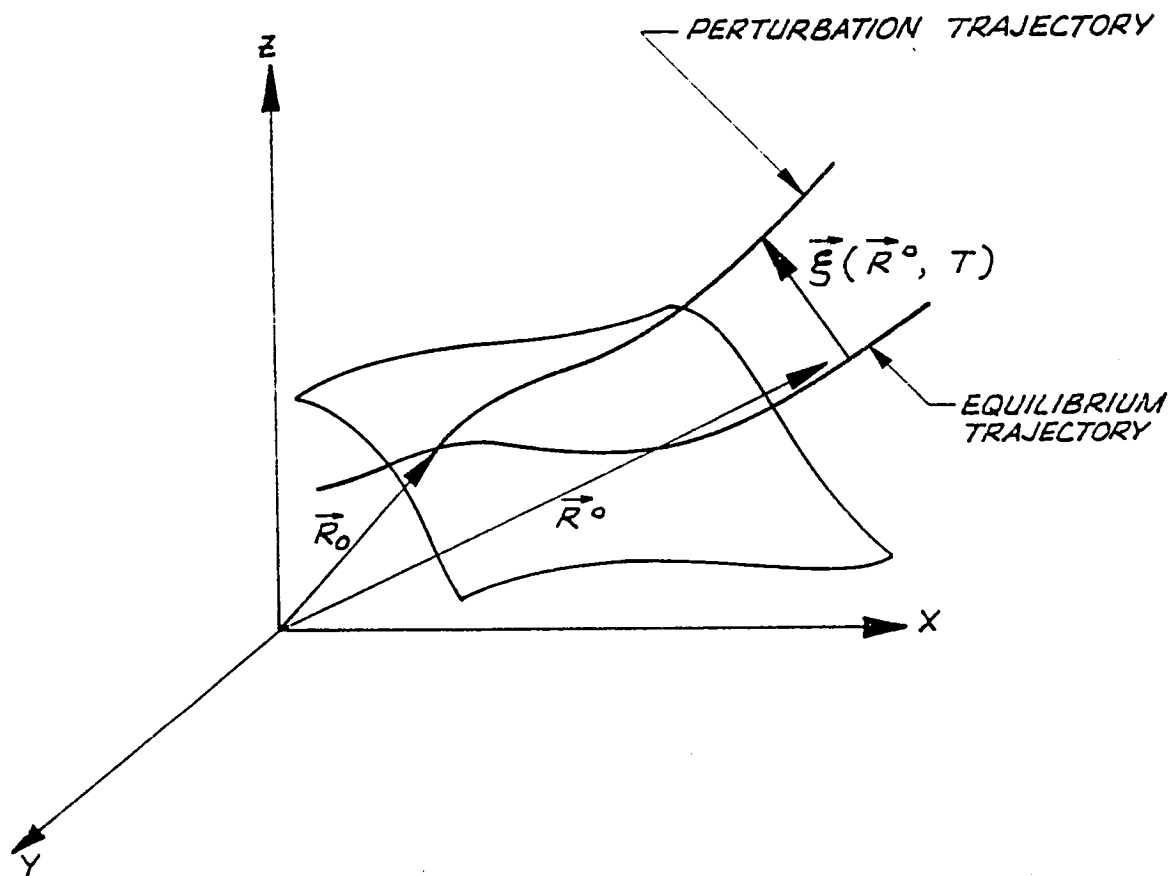   velocity component.

Fig. 5b. Effect of swirl on maximum
          circumferential velocity
          component.

CONFIGURATION STABILITY
(LAGRANGIAN FORMALISM)

$\vec{R}^0$ POSITION IN SPACE AT TIME T

$\vec{R}_0$ POSITION IN SPACE AT TIME T_0

$\vec{\xi}(\vec{R}^0, T)$ DISPLACEMENT VECTOR RELATIVE TO THE EQUILIBRIUM TRAJECTORY

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